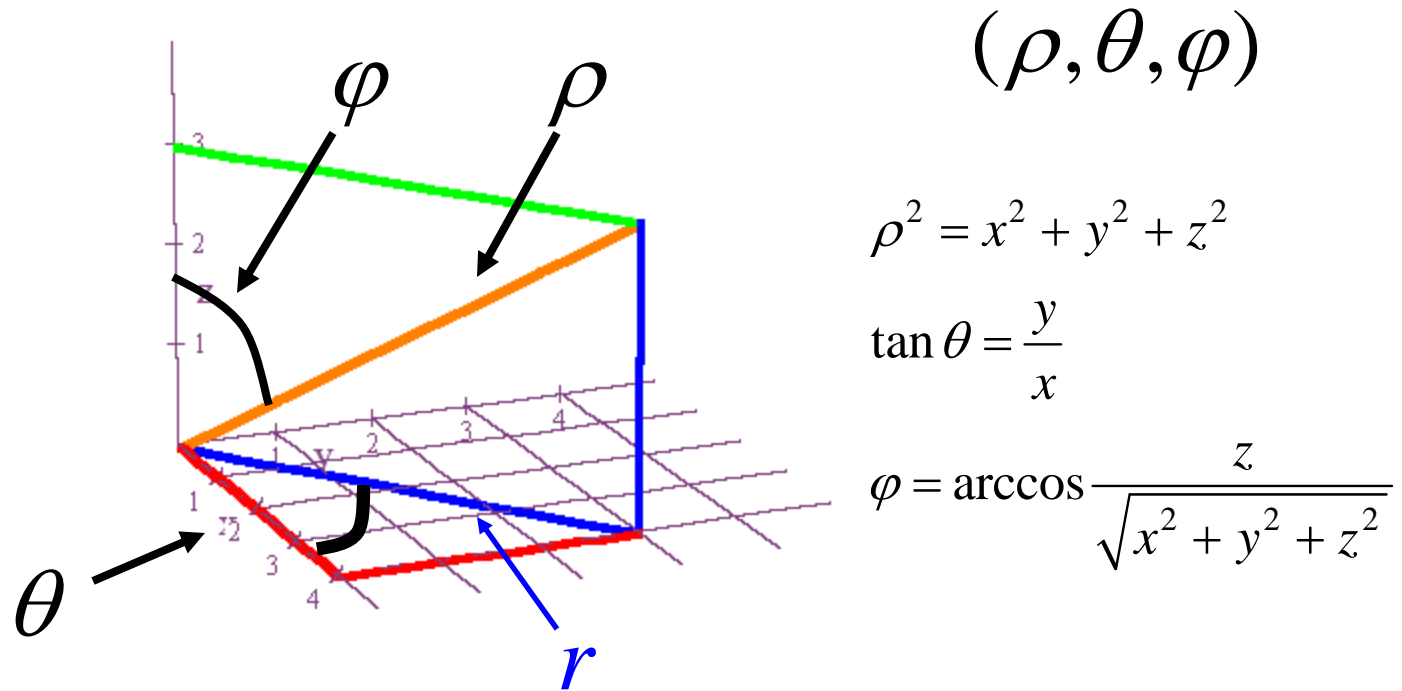


INTEGRALS IN SPHERICAL COORDINATES



$$x = r \cos(\theta) = \rho \sin(\varphi) \cos(\theta)$$

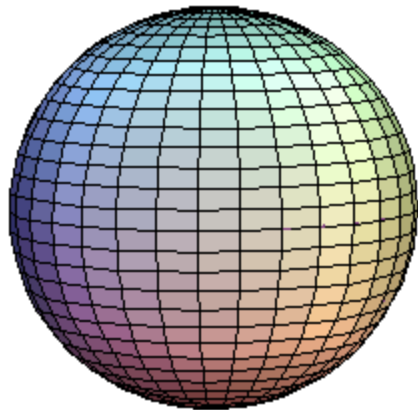
$$y = r \sin(\theta) = \rho \sin(\varphi) \sin(\theta)$$

$$z = \rho \cos(\varphi)$$

$$0 \leq \rho < \infty$$

$$0 \leq \varphi \leq \pi$$

$$0 \leq \theta < 2\pi$$

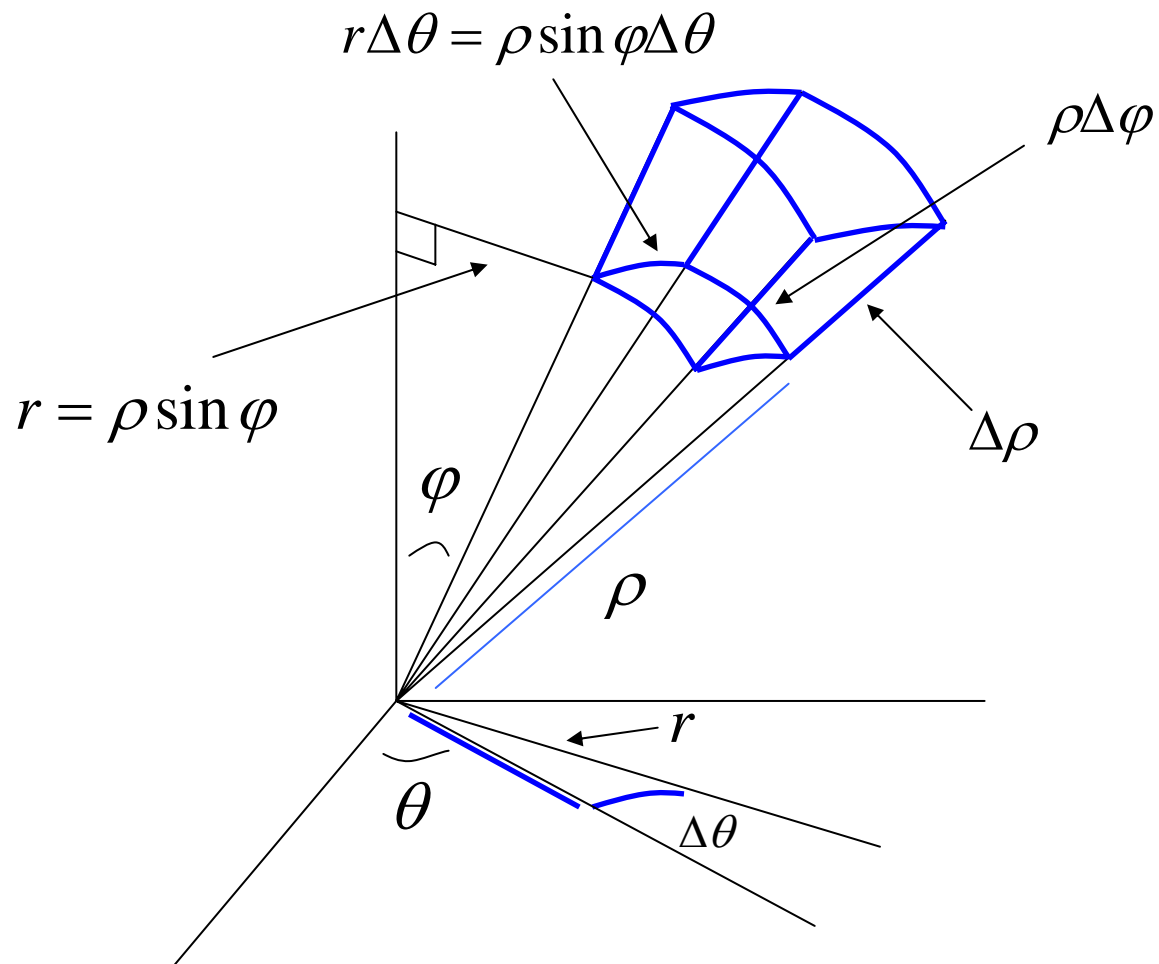


$$\rho = 5$$

$$0 \leq \theta < 2\pi$$

$$0 \leq \varphi \leq \pi$$

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

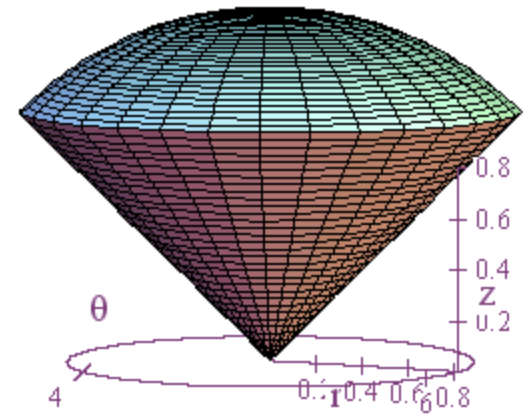


$$\Delta V \approx \Delta\rho \cdot \rho\Delta\varphi \cdot \rho\sin\varphi\Delta\theta = \rho^2 \sin\varphi \Delta\rho\Delta\varphi\Delta\theta$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

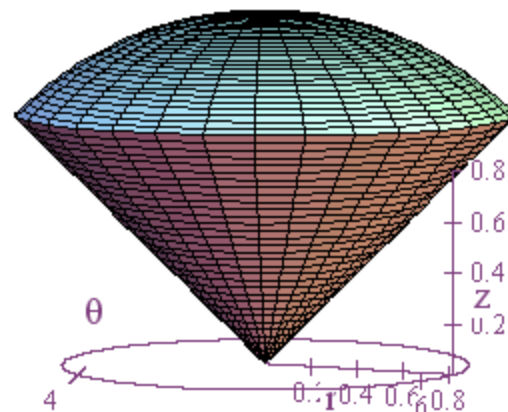
EXAMPLE: Find the volume of the ice cream cone defined by

$$0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{4}, 0 \leq \rho \leq \sqrt{2}.$$



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$$\begin{aligned} \text{Volume} &= \iiint_V dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin \varphi d\rho d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3} \sin \varphi \Big|_0^{\sqrt{2}} d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{2^{3/2}}{3} \sin \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \frac{2^{3/2}}{3} (-\cos \varphi) \Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{2^{3/2}}{3} \left(\frac{-1}{\sqrt{2}} + 1 \right) d\theta \\ &= \frac{\theta \cdot 2^{3/2}}{3} \left(\frac{-1}{\sqrt{2}} + 1 \right) \Big|_0^{2\pi} = \frac{2\pi}{3} \cdot 2^{3/2} \left(\frac{-1}{\sqrt{2}} + 1 \right) = \frac{4\pi}{3} \cdot \sqrt{2} \left(\frac{-1}{\sqrt{2}} + 1 \right) \\ &= \frac{4\pi}{3} (\sqrt{2} - 1) \end{aligned}$$

EXAMPLE: Find $\iiint_V z^2 dV$ on the region between

the spheres $\rho = 1$ and $\rho = 2$.

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

$$1 \leq \rho \leq 2$$

$$z = \rho \cos \varphi \Rightarrow z^2 = \rho^2 \cos^2 \varphi$$

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$$\iiint_V z^2 dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^2 \cos^2 \varphi \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$\int_0^{2\pi} \int_0^\pi \left(\frac{\rho^5}{5} \cos^2 \varphi \sin \varphi \right) \Big|_1^2 d\varphi d\theta = \int_0^{2\pi} \int_0^\pi \frac{31}{5} \cos^2 \varphi \sin \varphi d\varphi d\theta$$

$$= \int_0^{2\pi} \frac{31}{5} \frac{(-\cos^3 \varphi)}{3} \Big|_0^\pi d\theta = \int_0^{2\pi} \frac{62}{15} d\theta = \frac{62\theta}{15} \Big|_0^{2\pi}$$

$$= \frac{62 \cdot 2\pi}{15} - \frac{62 \cdot 0}{15} = \frac{124\pi}{15}$$