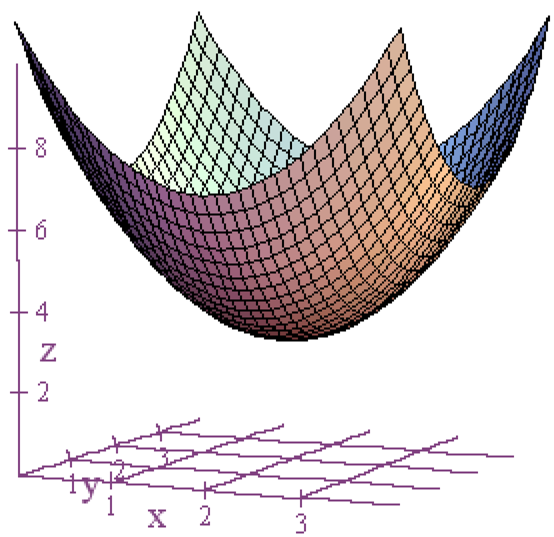
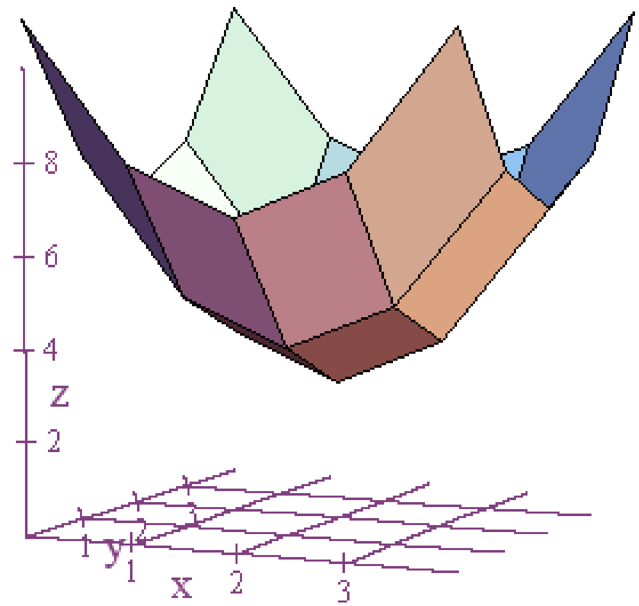
SURFACE AREA

Suppose we want to find the area of the surface below.
This can be done using double integrals!



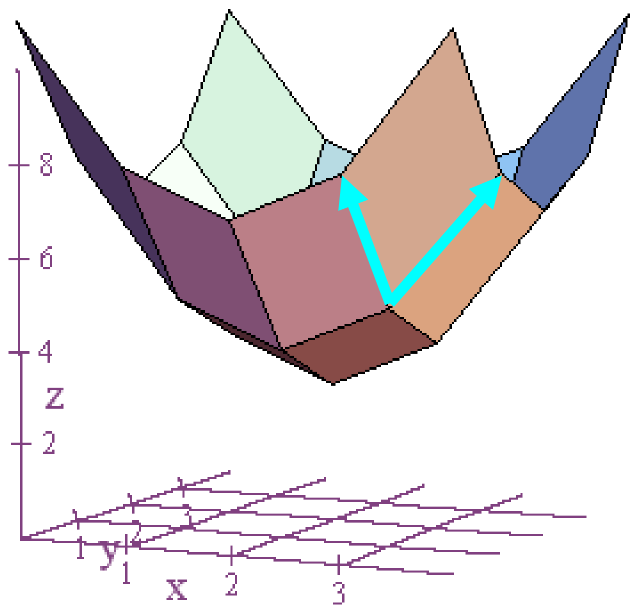
$$z = (x-2)^2 + (y-2)^2 + 3$$
$$0 \le x \le 4$$
$$0 \le y \le 4$$

First, imagine the smooth surface being approximated by a series of parallelograms.



$$z = (x-2)^2 + (y-2)^2 + 3$$
$$0 \le x \le 4$$
$$0 \le y \le 4$$

Each parallelogram on the surface is defined by a pair of vectors, *u* and *v*.

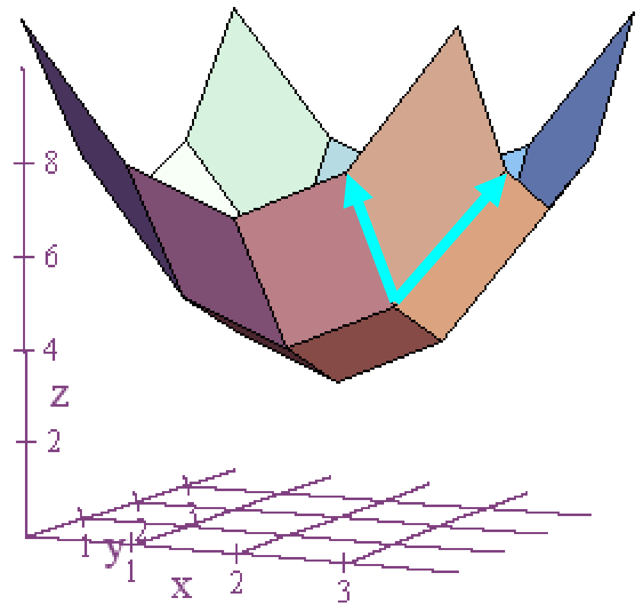


$$z = (x-2)^2 + (y-2)^2 + 3$$

$$0 \le x \le 4$$

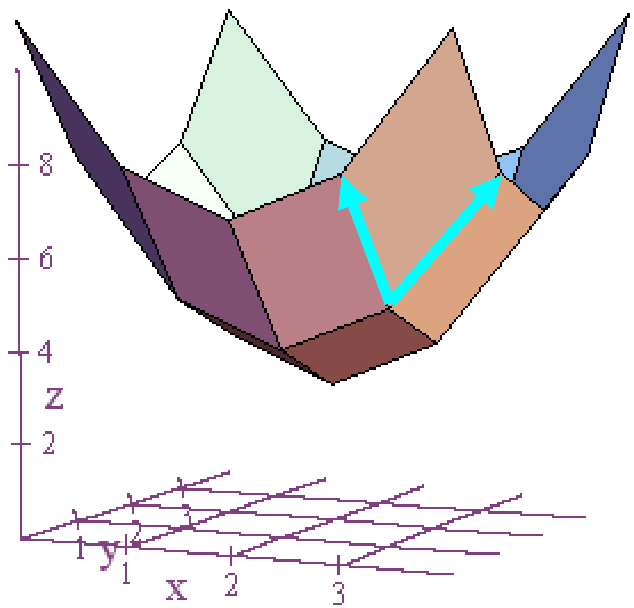
$$0 \le y \le 4$$

And directly below each parallelogram on the surface is a rectangle in the *xy*-plane.



$$z = (x-2)^2 + (y-2)^2 + 3$$
$$0 \le x \le 4$$
$$0 \le y \le 4$$

If we designate, on our rectangle in the *xy*-plane, the corner point of with the smallest coordinates as (x,y), then the two adjacent corner points will have coordinates (x+Δx,y) and (x,y+Δy).



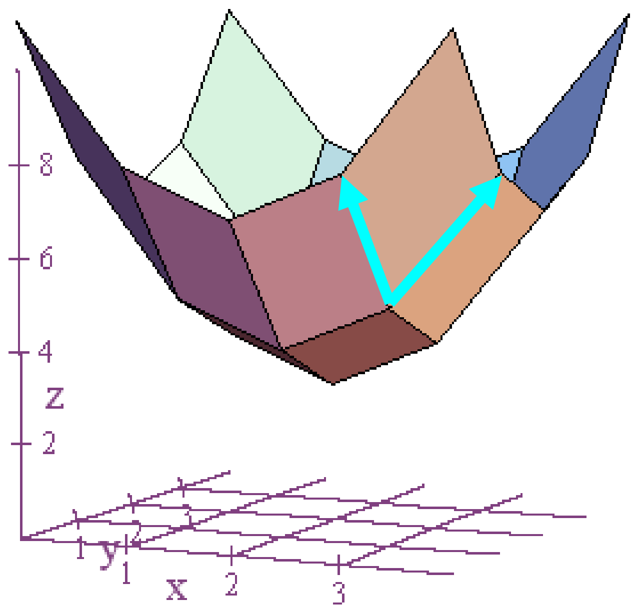
$$z = (x-2)^2 + (y-2)^2 + 3$$

$$0 \le x \le 4$$

$$0 \le y \le 4$$

I claim we can define our vectors *u* and *v* as,

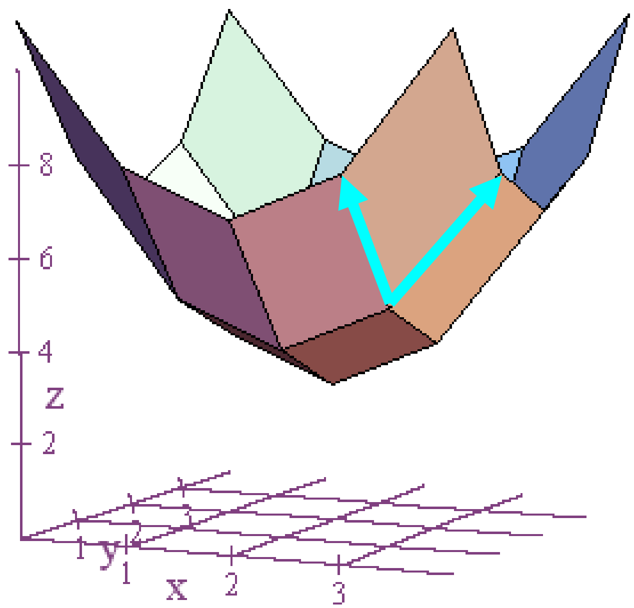
$$\vec{u} \approx \Delta x\,\hat{i} + 0\,\hat{j} + \frac{\partial f(x,y)}{\partial x}\Delta x\,\hat{k} \qquad \vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \frac{\partial f(x,y)}{\partial y}\Delta y\,\hat{k}$$



$$z = (x-2)^2 + (y-2)^2 + 3$$
$$0 \le x \le 4$$
$$0 \le y \le 4$$

Think about why this works!

$$\vec{u} \approx \Delta x\,\hat{i} + 0\,\hat{j} + \frac{\partial f(x,y)}{\partial x}\Delta x\,\hat{k} \qquad \vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \frac{\partial f(x,y)}{\partial y}\Delta y\,\hat{k}$$



$$z = (x-2)^2 + (y-2)^2 + 3$$
$$0 \le x \le 4$$
$$0 \le y \le 4$$

We can now find the area of a parallelogram using one of the formulas we developed earlier.

$$\vec{u} \approx \Delta x \,\hat{i} + 0\,\hat{j} + \frac{\partial f(x,y)}{\partial x}\Delta x\,\hat{k} \qquad \vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \frac{\partial f(x,y)}{\partial y}\Delta y\,\hat{k}$$

$$\vec{u}\times\vec{v} \approx \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x}\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}\Delta y \end{vmatrix} = \left(-\frac{\partial f}{\partial x}\Delta x\Delta y\right)\hat{i} - \left(\frac{\partial f}{\partial y}\Delta x\Delta y\right)\hat{j} + \left(\Delta x\Delta y\right)\hat{k}$$

And,

$$\left\|\vec{u}\times\vec{v}\right\| \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \Delta x^2\Delta y^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta x^2\Delta y^2 + \Delta x^2\Delta y^2} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\cdot\Delta x\Delta y$$

Therefore,

$$Surface\ Area = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta x \Delta y$$

$$= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dxdy = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

If we let *S* denote the surface we are integrating over, and if we denote an element of area on the surface by ΔS, then what we've also shown above is that,

$$\Delta S \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta A$$

And,

$$Surface\ Area = \iint_S dS = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

In differential form, we write this as,

$$dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$$

Unfortunately, for many surface integrals we wind up with something very difficult to integrate by hand. Here is what we get for our example problem.

$$z = (x-2)^2 + (y-2)^2 + 3$$

$$0 \le x \le 4$$

$$0 \le y \le 4$$

$$z_x = 2(x-2) = 2x - 4$$

$$z_y = 2(y-2) = 2y - 4$$

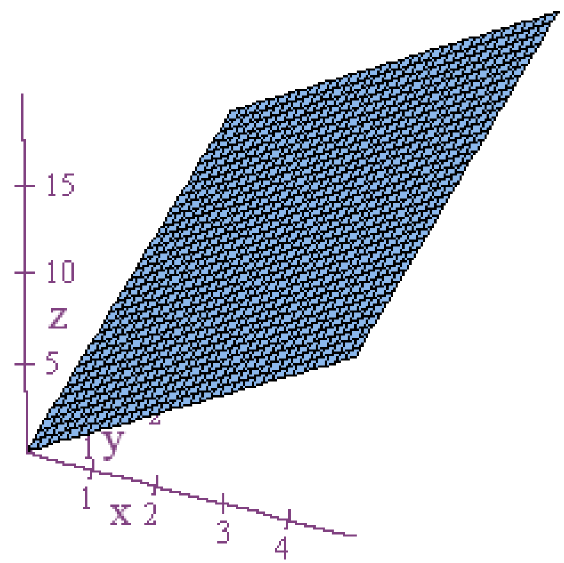
$$z_x^{\ 2} = 4x^2 - 16x + 16$$

$$z_y^{\ 2} = 4y^2 - 16y + 16$$

$$Surface\ Area = \iint_R \sqrt{4x^2 + 4y^2 - 16x - 16y + 33}\, dA$$

However, if our surface is a plane, then it’s easy to do.

$$z = 2x + 3y$$
$$0 \le x \le 5$$
$$0 \le y \le 4$$



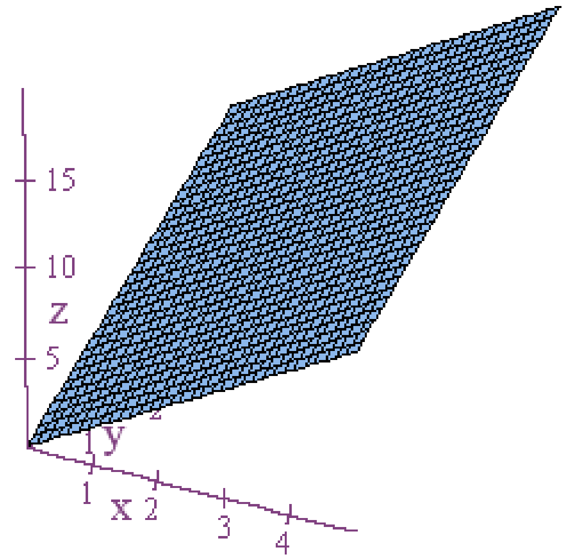
$$z = 2x + 3y$$

$$0 \le x \le 5$$

$$0 \le y \le 4$$

$$z_x = 2$$

$$z_y = 3$$



$$\sqrt{z_x^{\,2} + z_y^{\,2} + 1} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$Surface\ Area = \iint_S dS = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

$$\int_0^5 \int_0^4 \sqrt{14}\, dy dx = \int_0^5 y\sqrt{14} \Big|_0^4 dx = \int_0^5 4\sqrt{14}\, dx = 4x\sqrt{14} \Big|_0^5 = 20\sqrt{14} \approx 74.833$$

We can verify this result by finding the vectors
u and *v* and calculating directly the area of a parallelogram.

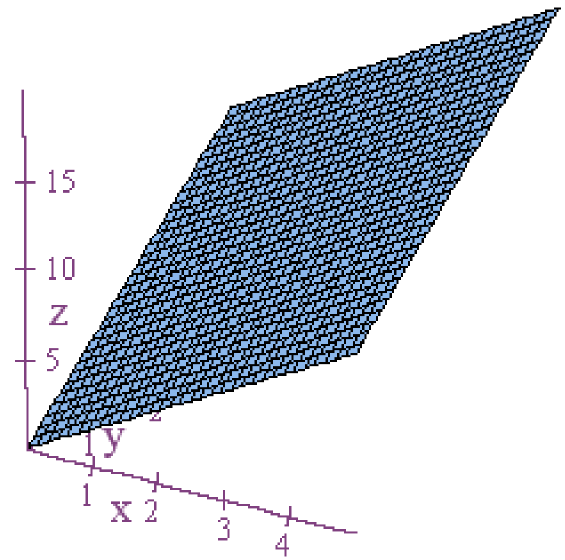
$$z = 2x + 3y$$

$$0 \le x \le 5$$

$$0 \le y \le 4$$

$$\vec{u} = 5\hat{i} + +10\hat{k}$$

$$\vec{v} = 4\hat{j} + 12\hat{k}$$



$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 10 \\ 0 & 4 & 12 \end{vmatrix} = -40\hat{i} - 60\hat{j} + 20\hat{k}$$

$$\left\| \vec{u} \times \vec{v} \right\| = \sqrt{1600 + 3600 + 400} = \sqrt{5600} = \sqrt{400 \cdot 14} = 20\sqrt{14} \approx 74.833$$