Triple Integrals

Change in volume $= \Delta V = \Delta x \Delta y \Delta z$

$dV = dxdydz$

Let $w = f(x, y, z)$

$$\iiint_V f(x,y,z)\,dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(x_{ijk}, y_{ijk}, z_{ijk})\,\Delta V$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdydx$$

Volume $= \iiint_V dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} dzdydx$

EXAMPLE: Find the volume of the solid in the first octant that is bounded below by the xy-plane and above by $x+y+z=1$.

$$0 \le x \le 1$$

$$0 \le y \le -x+1$$

$$0 \le z \le 1-x-y$$

$$\text{Volume } = \iiint_V dV = \int_0^1 \int_0^{-x+1} \int_0^{1-x-y} dz\,dy\,dx = \int_0^1 \int_0^{-x+1} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Bigg|_0^{-x+1} dx = \int_0^1 \left(-x+1-x(-x+1) - \frac{(-x+1)^2}{2} \right) dx$$

$$= \int_0^1 \left(-x+1+x^2-x-\frac{x^2-2x+1}{2} \right) dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx$$

$$= \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \right) \Bigg|_0^1 = \frac{1}{6}$$

EXAMPLE: Let R be the solid region between the graphs of $z=-y^2$ and $z=x^2$ where $0\le x\le 1$ and $0\le y\le x$. Evaluate $\iiint_R (x+1)\,dV$.

$$0\le x\le 1$$
$$0\le y\le x$$
$$-y^2\le z\le x^2$$

$$\iiint_V (x+1)\,dV=\int_0^1\int_0^x\int_{-y^2}^{x^2}(x+1)\,dzdydx=\int_0^1\int_0^x z(x+1)\Big|_{-y^2}^{x^2}dydx$$

$$=\int_0^1\int_0^x (x^3+x^2)+(xy^2+y^2)\,dydx=\int_0^1\left(x^3y+x^2y+\frac{xy^3}{3}+\frac{y^3}{3}\right)\Bigg|_0^x dx$$

$$=\int_0^1\left(x^4+x^3+\frac{x^4}{3}+\frac{x^3}{3}\right)dx=\int_0^1\left(\frac{4x^4}{3}+\frac{4x^3}{3}\right)dx=\left(\frac{4x^5}{15}+\frac{4x^4}{12}\right)\Bigg|_0^1$$

$$=\frac{4}{15}+\frac{4}{12}=\frac{16}{60}+\frac{20}{60}=\frac{36}{60}=\frac{6}{10}=\frac{3}{5}$$

<u>NOTE:</u> In physics, triple integrals are used to compute other things besides just volume.

Let $\rho = \rho(x, y, z) \dfrac{\text{mass}}{\text{volume}}$ be a density function.

1. Mass $= \iiint_V p \, dV$

2. First moment about yz-plane $= \iiint_V xp \, dV$

3. First moment about xz-plane $= \iiint_V yp \, dV$

4. First moment about xy-plane $= \iiint_V zp \, dV$

5. Center of mass $= \left(\overline{x}, \overline{y}, \overline{z}\right) = \left(\dfrac{\iiint_V xp \, dV}{\iiint_V p \, dV}, \dfrac{\iiint_V yp \, dV}{\iiint_V p \, dV}, \dfrac{\iiint_V zp \, dV}{\iiint_V p \, dV} \right)$

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Let $\rho = \rho(x, y, z) \dfrac{\text{mass}}{\text{volume}}$ be a density function.

6. Moment of inertia about the x-axis $= \iiint_V (y^2 + z^2)\, p\, dV$

7. Moment of inertia about the y-axis $= \iiint_V (x^2 + z^2)\, p\, dV$

8. Moment of inertia about the z-axis $= \iiint_V (x^2 + y^2)\, p\, dV$

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The Joy of Physics!