The Dot Product
&
The Cross Product

Mathematicians are often a little weird.

They are frequently obsessive-compulsive.

Give them a new toy, and they immediately
want to do arithmetic with it.

At one point, their new toy was “vectors.”

We’ve seen how to add and subtract vectors, …

But how do you multiply vectors?

In 1773, Joseph Lagrange was working on a problem involving tetrahedrons, and he came up with two ways to multiply vectors.

Today, we call these two methods the “dot product” and the “cross product.”

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Ideas are also subject to the doctrine of survival of the fittest.

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a. Extremely useful, and

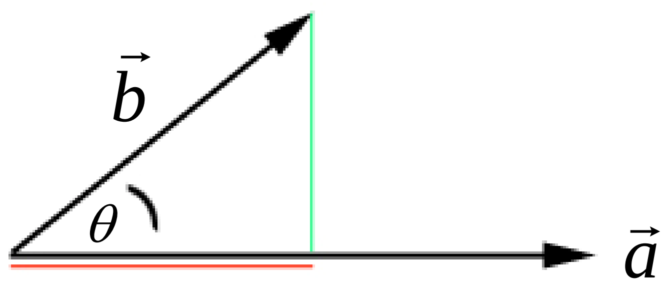
b. Rather bizarre in appearance.

The Dot Product

We’ll define the dot product as follows:

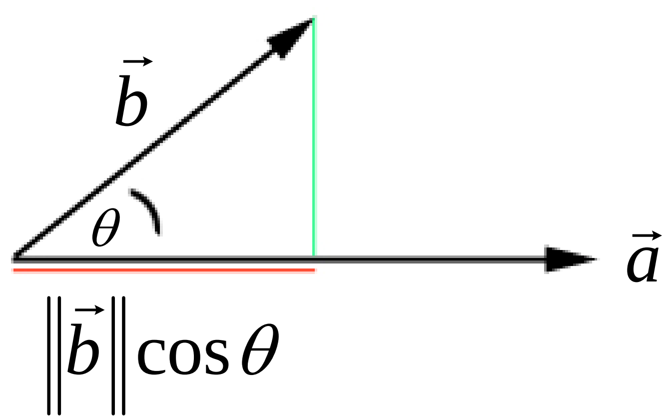
$$If\ \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ and\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

$$then\ \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta,\ where\ \theta\ is\ the\ angle\ between\ the\ vectors.$$



What does this definition mean?

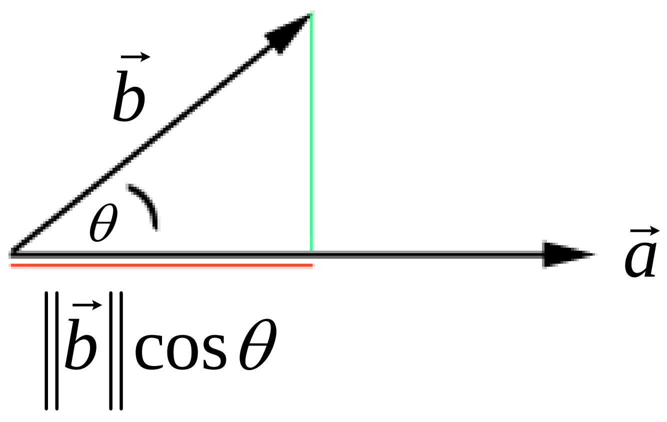
$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



What does this definition mean?

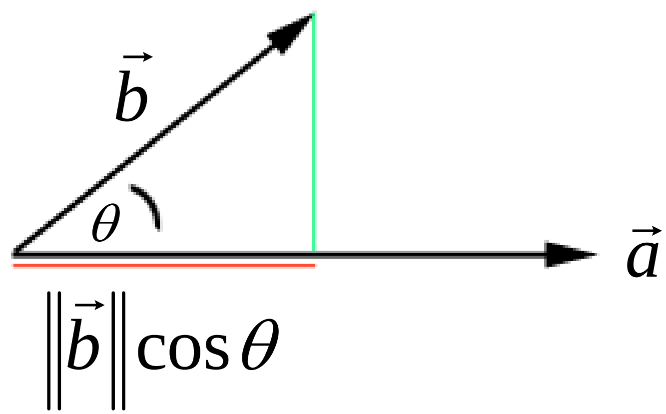
We can think of the dot product as the length of vector *a* times the length of the component of vector *b* that is parallel to vector *a.*

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



Alternatively, we could think of it as the length of vector *b* times the length of the component of vector *a* that is parallel to vector *b.*

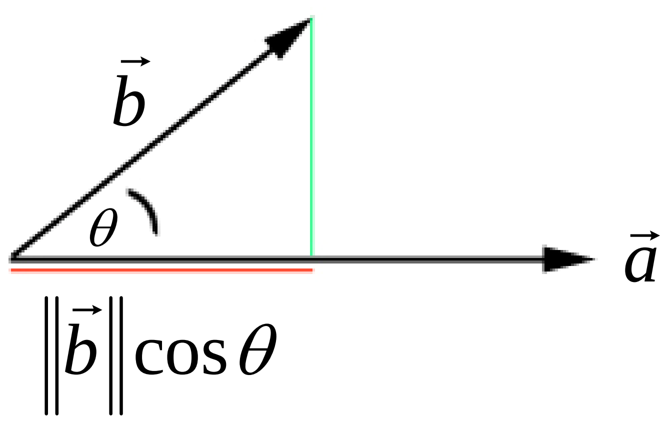
$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



Alternatively, we could think of it as the length of vector *b* times the length of the component of vector *a* that is parallel to vector *b.*

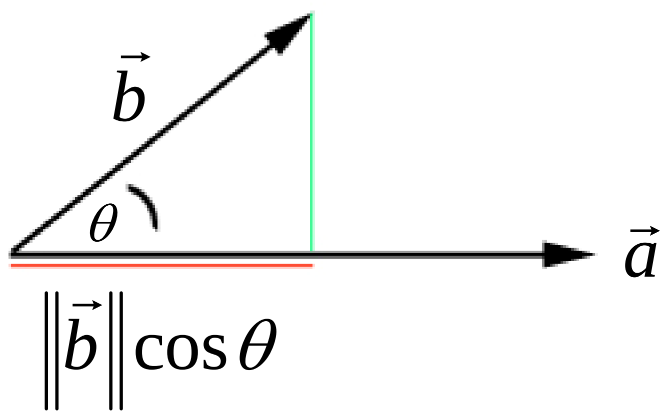
The result is the same either way.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



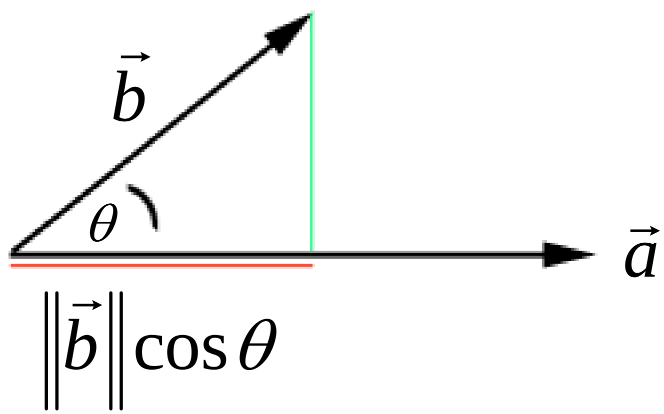
The dot product gives us a way of multiplying two vectors such that the result is a number (or scalar), not another vector.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



The problem, though, is that we need to know the angle between the vectors in order to accomplish this.

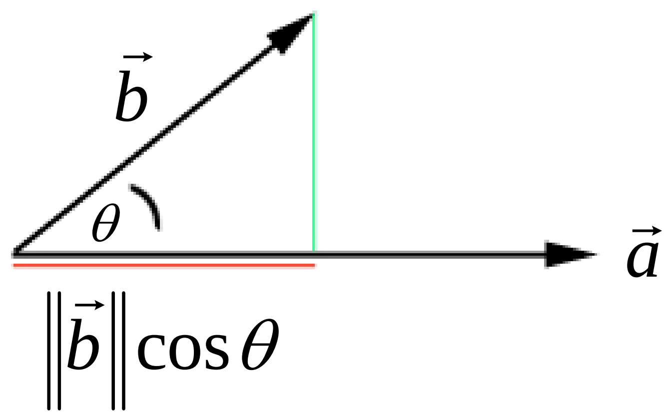
$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$



The problem, though, is that we need to know the Angle between the vectors in order to accomplish This.

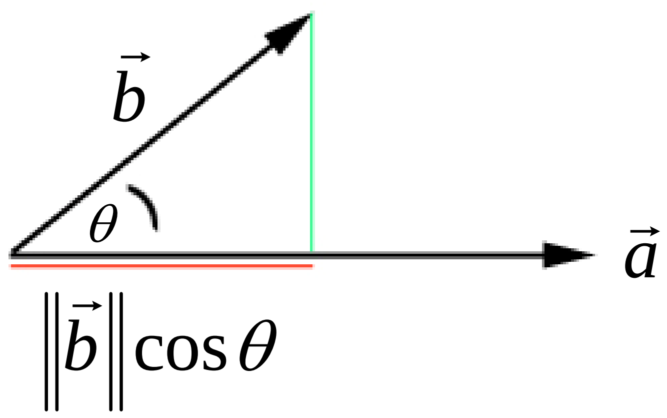
And this is something we often don't know.

$$\vec{a} \cdot \vec{b} = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$$



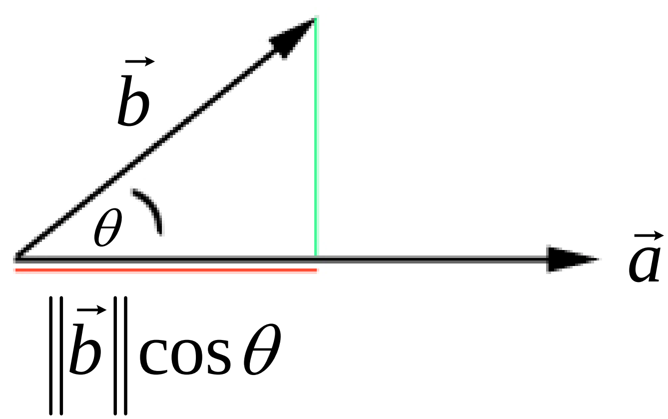
Fortunately, however, we have a theorem to help us.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



$$Theorem: \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

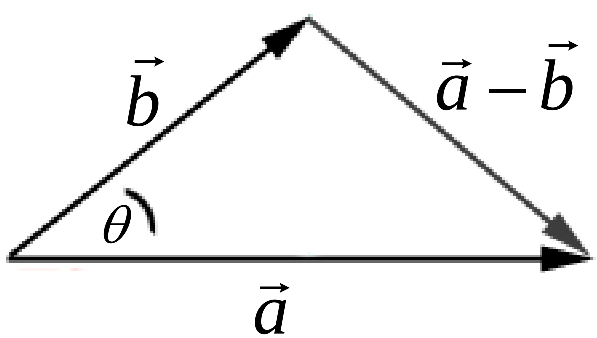
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$



$$Theorem: \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Proof: By the law of cosines,

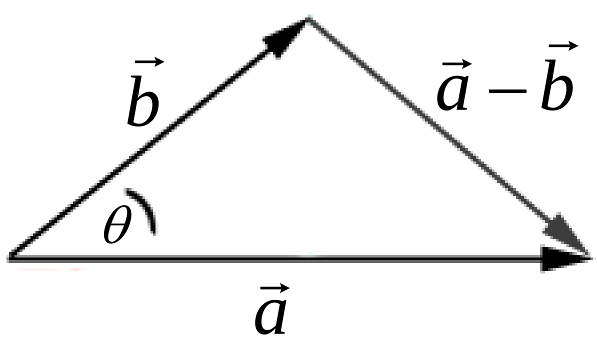
$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$$



But this implies that,

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$

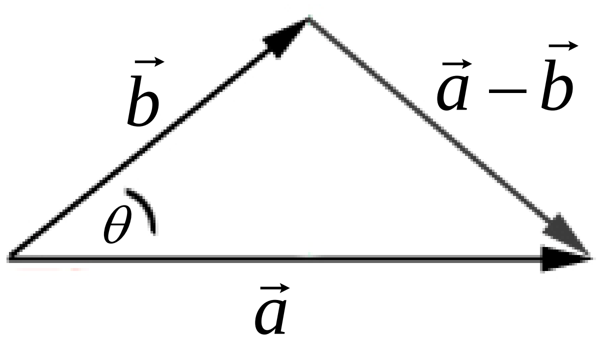
$$= a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$$



Expanding the left side of the equation yields,

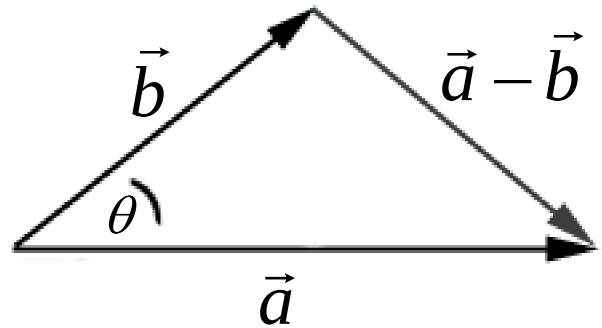
$$a_1^{\,2} - 2a_1b_1 + b_1^{\,2} + a_2^{\,2} - 2a_2b_2 + b_2^{\,2} + a_3^{\,2} - 2a_3b_3 + b_3^{\,2}$$

$$= a_1^{\,2} + a_2^{\,2} + a_3^{\,2} + b_1^{\,2} + b_2^{\,2} + b_3^{\,2} - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta.$$



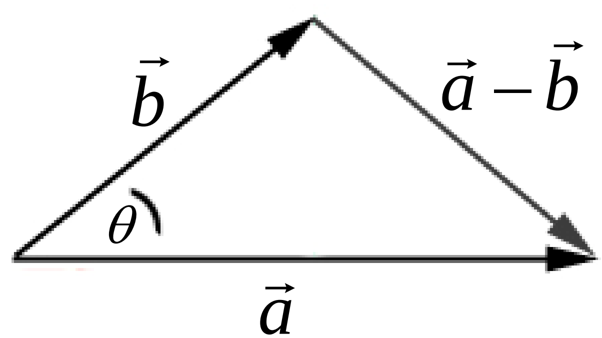
Next, subtracting like terms from each side gives,

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta.$$



And finally, we divide each side by -2.

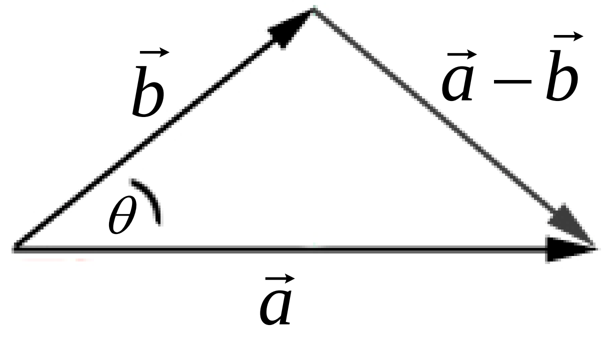
$$a_1b_1 + a_2b_2 + a_3b_3 = \|\vec{a}\|\|\vec{b}\|\cos\theta = \vec{a}\cdot\vec{b}.$$



As a corollary, we get the following formula for finding the angle between two vectors.

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\vec{a}\|\|\vec{b}\|} = \cos\theta.$$

$$0 \le \theta \le \pi$$



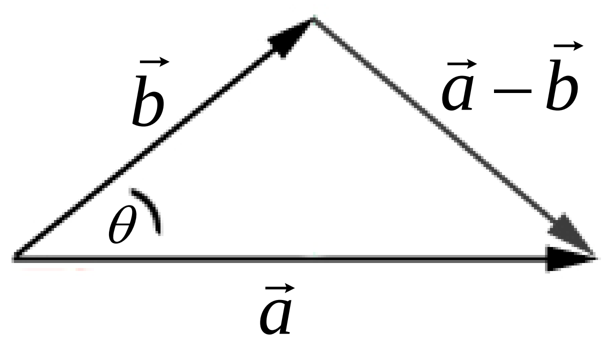
EXAMPLE:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = (2)(-1) + (3)(2) + (4)(-2) = -4$$

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-4}{\sqrt{4+9+16}\sqrt{1+4+4}} = \frac{-4}{3\sqrt{29}} = \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{3\sqrt{29}}\right) \approx 104.3°$$



A fairly immediate and important consequence of what we've seen is the following:

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = 0 \textit{ if and only if } \theta = 90^\circ \textit{ or } \vec{a} = \vec{0} \textit{ or } \vec{b} = \vec{0}.$$

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As a final note, the dot product is also known as the "scalar product."

The Cross Product

The cross product of two vectors is defined as follows:

$$If\ \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ and\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

$$then\ \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$$= (a_2b_3 - a_3.b_2)\hat{i} - (a_1b_3 - a_3.b_1)\hat{j} + (a_1b_2 - a_2.b_1)\hat{k}.$$

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Isn’t that cool!

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$$\vec{a}\cdot\left(\vec{a}\times\vec{b}\right) = a_1(a_2b_3 - a_3.b_2) + a_2\left[-(a_1b_3 - a_3.b_1)\right] + a_3(a_1b_2 - a_2.b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 = 0$$

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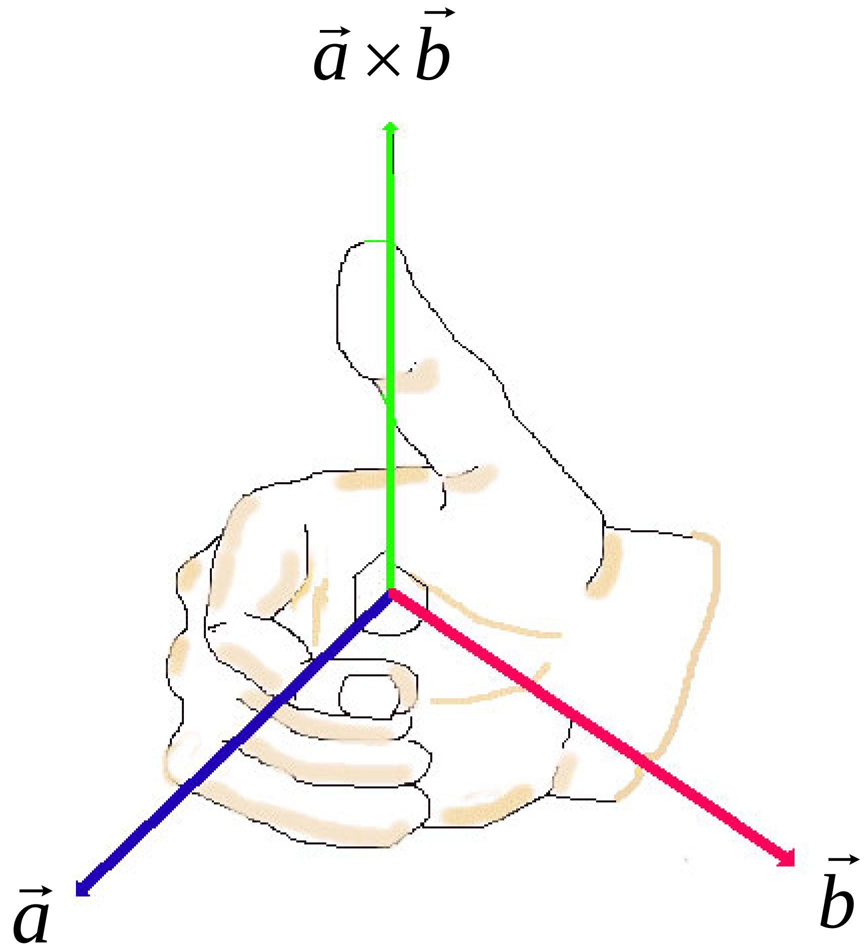
The proof that the cross product is perpendicular to vector *b* is similar.

The cross product, $\vec{a} \times \vec{b}$, obeys a right-hand rule.



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One of the more important ones is below:

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The proof is amazingly simple!

<u>Proof:</u>

$$\left\|\vec{a}\times\vec{b}\right\|^2 = (a_2b_3 - a_3.b_2)^2 + (a_1b_3 - a_3.b_1)^2 + (a_1b_2 - a_2.b_1)^2$$

$$= a_2^{\ 2}b_3^{\ 2} - 2a_2a_3b_2b_3 + a_3^{\ 2}b_2^{\ 2} + a_1^{\ 2}b_3^{\ 2} - 2a_1a_3b_1b_3 + a_3^{\ 2}b_1^{\ 2}$$

$$+ a_1^{\ 2}b_2^{\ 2} - 2a_1a_2b_1b_2 + a_2^{\ 2}b_1^{\ 2}$$

$$= a_1^{\ 2}b_2^{\ 2} + a_1^{\ 2}b_3^{\ 2} + a_2^{\ 2}b_1^{\ 2} + a_2^{\ 2}b_3^{\ 2} + a_3^{\ 2}b_1^{\ 2} + a_3^{\ 2}b_2^{\ 2}$$

$$- 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3$$

$$= a_1^{\ 2}b_1^{\ 2} + a_2^{\ 2}b_2^{\ 2} + a_3^{\ 2}b_3^{\ 2} + a_1^{\ 2}b_2^{\ 2} + a_1^{\ 2}b_3^{\ 2} + a_2^{\ 2}b_1^{\ 2} + a_2^{\ 2}b_3^{\ 2} + a_3^{\ 2}b_1^{\ 2} + a_3^{\ 2}b_2^{\ 2}$$

$$- 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 - a_1^{\ 2}b_1^{\ 2} - a_2^{\ 2}b_2^{\ 2} - a_3^{\ 2}b_3^{\ 2}$$

$$= (a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2})(b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2}) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2 - \left(\vec{a}\cdot\vec{b}\right)^2 = \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2 - \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\right)^2$$

$$= \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2 - \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2\cos^2\theta = \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2\left(1-\cos^2\theta\right) = \left\|\vec{a}\right\|^2\left\|\vec{b}\right\|^2\sin^2\theta.$$

Thus,

$$\left\|\vec{a}\times\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 \sin^2\theta.$$

And since,

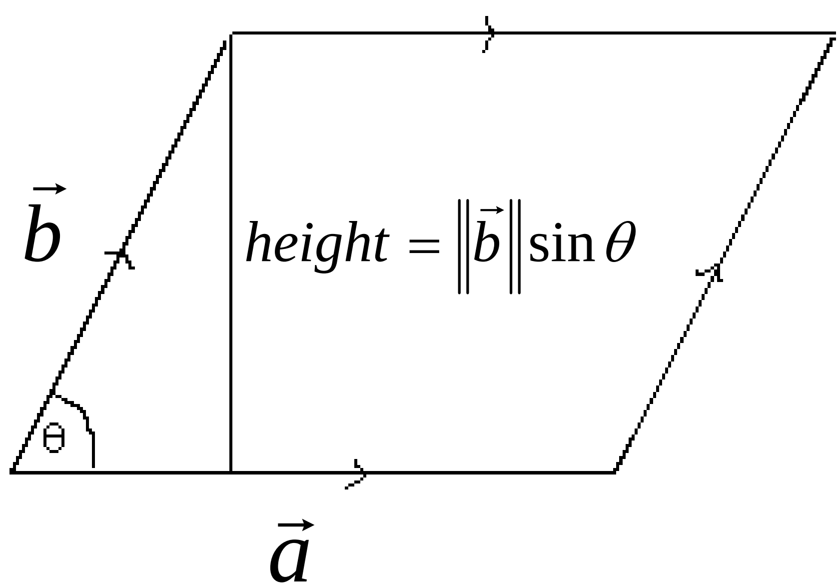
$$\sin\theta \geq 0 \;\; for\, 0 \leq \theta \leq \pi,$$

We have that,

$$\left\|\vec{a}\times\vec{b}\right\| = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \sin\theta.$$

One application of this result is a formula for the area of a parallelogram.

$$Area = \|\vec{a}\|\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|$$

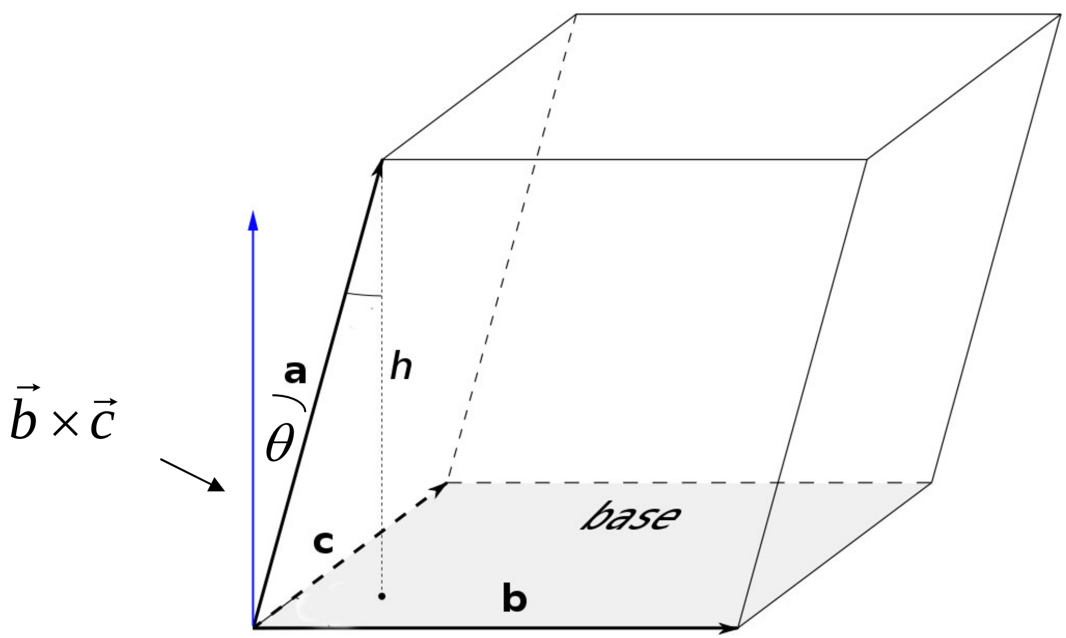


Another result is a formula for the volume of a parallelepiped.

$$Area\,of\ base = \left\|\vec{b} \times \vec{c}\right\|$$

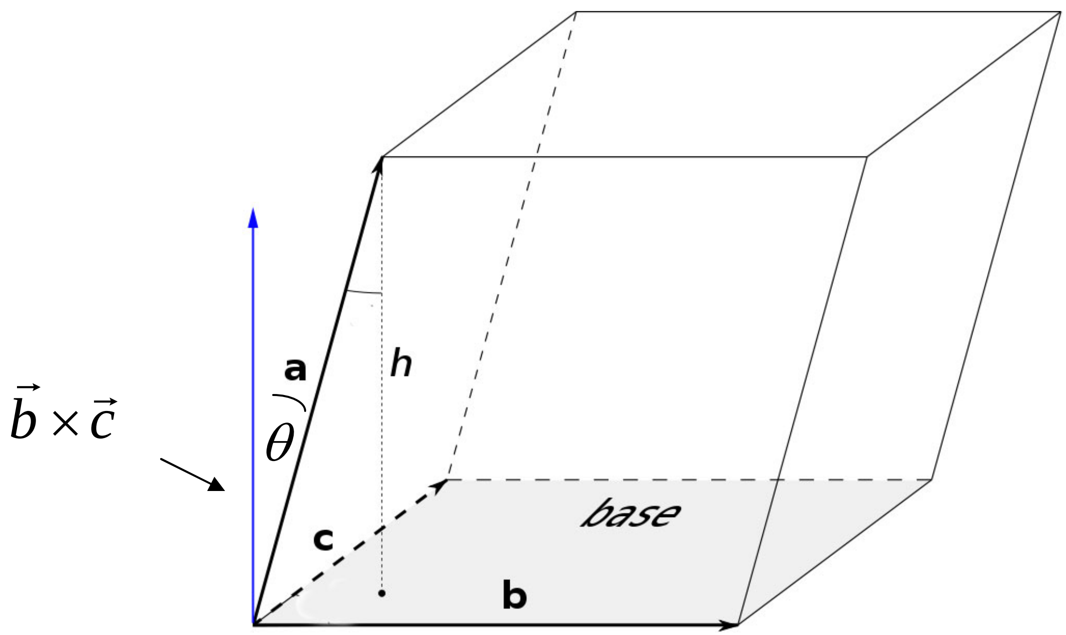
$$height = \left\|\vec{a}\right\| \cos\theta,$$

where θ is the acute angle between *b x c* and *a*.



This suggests the following formula for the volume.

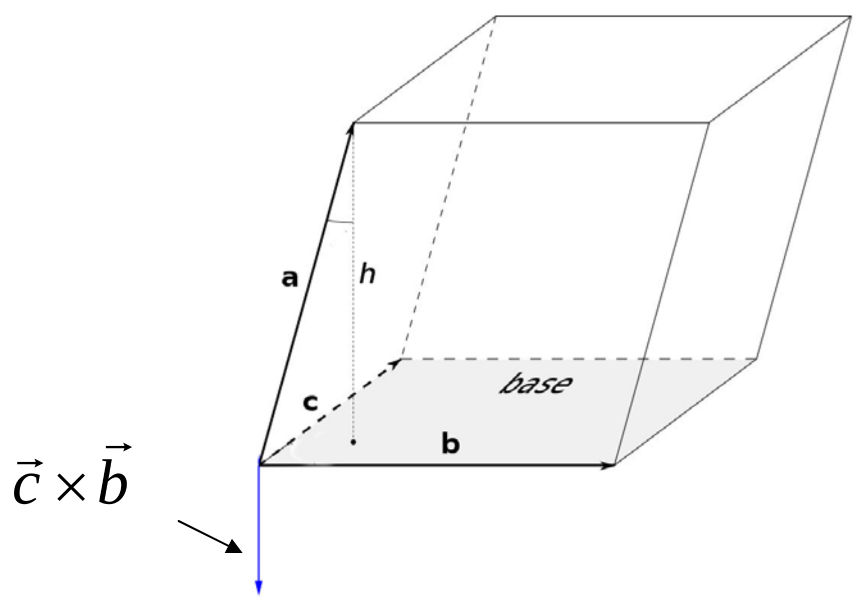
$$Volume = Area\,of\ base \times height = \left\|\vec{b} \times \vec{c}\right\| \cdot \left\|\vec{a}\right\| \cos\theta = \left(\vec{b} \times \vec{c}\right) \bullet \vec{a}$$



However, if we don't take our cross product in the right order, then cosθ will be negative. The fix is to just take the absolute value of the whole thing.

$$Volume =$$

$$\left|\left(\vec{c}\times\vec{b}\right)\bullet\vec{a}\right| = \left|\left(\vec{b}\times\vec{c}\right)\bullet\vec{a}\right| = \left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right| = absolute\ value\ of\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



EXAMPLE (Volume of Parallelepiped):

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$$

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$$Volume = \left| \vec{a} \bullet \left(\vec{b} \times \vec{c} \right) \right| = absolute\,value\,of \begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & -2 \\ -1 & -3 & 4 \end{vmatrix} = 42$$

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$$\vec{a} \bullet \left(\vec{b} \times \vec{c}\right) \text{ is also known as the scalar triple product.}$$

<u>EXAMPLE (Cross Product):</u>

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$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ -1 & 2 & -2 \end{vmatrix} = -14\hat{i} + 0\hat{j} + 7\hat{k} = -14\hat{i} + 7\hat{k}$$

EXAMPLE (Area of aParallelogram):

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EXAMPLE (Area of aParallelogram):

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$
$$\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$$

$$Area = \left\|\vec{a} \times \vec{b}\right\| = \left\|-14\hat{i} + 7\hat{k}\right\| = \sqrt{245} = 7\sqrt{5} \approx 15.65$$