SECOND PARTIALS TEST - ANSWERS

(1-7) For each of the functions below, find the critical points and the determinant of the second partials matrix, and classify each critical point as resulting in a local maximum, local minimum, saddle point, or inconclusive. Furthermore, for each critical point (a,b), specify the coordinates $\left(a,b,f(a,b)\right)$.

1. $z = f\left(x,y\right) = x^2 + y^2$

$z_x = 2x$
$z_y = 2y$

$$\begin{matrix} 2x = 0 \\ 2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = \left(0,0\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\left.\begin{matrix} D(0,0) = 4 > 0 \\ z_{xx}(0,0) = 2 > 0 \end{matrix}\right\} \Rightarrow \left(0,0,0\right) \text{ is a local minimum point}$$

2. $z = f\left(x,y\right) = x^2 - y^2$

$z_x = 2x$
$z_y = -2y$

$$\begin{matrix} 2x = 0 \\ -2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = \left(0,0\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

$$D(0,0) = -4 < 0 \Rightarrow \left(0,0,0\right) \text{ is a saddle point}$$

3. $z = f(x,y) = -\left(x^2 + y^2\right)$

$$z_x = -2x$$
$$z_y = -2y$$

$$\begin{matrix} -2x = 0 \\ -2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = (0,0)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$\left.\begin{matrix} D(0,0) = 4 > 0 \\ z_{xx}(0,0) = -2 < 0 \end{matrix}\right\} \Rightarrow (0,0,0) \text{ is a local maximum point}$$

4. $z = f(x,y) = x^3 - 6x + y^3 - 9y$

$$z_x = 3x^2 - 6 = 3(x^2 - 2)$$
$$z_y = 3y^2 - 9 = 3(y^2 - 3)$$

$$\begin{matrix} 3(x^2 - 2) = 0 \\ 3(y^2 - 3) = 0 \end{matrix} \Rightarrow \begin{matrix} x = \pm\sqrt{2} \\ y = \pm\sqrt{3} \end{matrix}$$
$$\Rightarrow \text{ critical point } = \left(-\sqrt{2},-\sqrt{3}\right),\left(\sqrt{2},\sqrt{3}\right),\left(-\sqrt{2},\sqrt{3}\right),\left(\sqrt{2},-\sqrt{3}\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$$

$$\left.\begin{matrix} D(-\sqrt{2},-\sqrt{3}) = 36\sqrt{6} > 0 \\ z_{xx}(-\sqrt{2},-\sqrt{3}) = -6\sqrt{2} < 0 \end{matrix}\right\} \Rightarrow \left(-\sqrt{2},-\sqrt{3},\approx 16.0492\right) \text{ is a local maximum point}$$

$$\left.\begin{matrix} D(\sqrt{2},\sqrt{3}) = 36\sqrt{6} > 0 \\ z_{xx}(\sqrt{2},\sqrt{3}) = 6\sqrt{2} > 0 \end{matrix}\right\} \Rightarrow \left(\sqrt{2},\sqrt{3},\approx -16.0492\right) \text{ is a local minimum point}$$

$$D(-\sqrt{2},\sqrt{3}) = -36\sqrt{6} < 0 \Rightarrow \left(-\sqrt{2},\sqrt{3},\approx -4.7355\right) \text{ is a saddle point}$$

$$D(\sqrt{2},-\sqrt{3}) = -36\sqrt{6} < 0 \Rightarrow \left(\sqrt{2},-\sqrt{3},\approx 4.7355\right) \text{ is a saddle point}$$

5. $z = f(x,y) = x^3 - 12xy - y^4$

$$z_x = 3x^2 - 12y = 3(x^2 - 4y)$$
$$z_y = -4y^3 - 12x = -4(y^3 + 3x)$$

$$\begin{aligned} 3(x^2-4y)&=0 \\ -4(3x+y^3)&=0 \end{aligned} \Rightarrow \begin{aligned} x^2-4y&=0 \\ 3x+y^3&=0 \end{aligned} \Rightarrow x = \frac{-y^3}{3} \Rightarrow x^2 = \frac{y^6}{9} \Rightarrow \frac{y^6}{9} - 4y = 0$$

$$\Rightarrow y^6 - 36y = 0 \Rightarrow y(y^5 - 36) = 0 \Rightarrow y = 0 \text{ or } y = 36^{1/5}$$

$$\Rightarrow \text{ critical point } = (0,0), \left(-\frac{36^{3/5}}{3}, 36^{1/5}\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -12 \\ -12 & -12y^2 \end{vmatrix} = -72xy^2 - 144$$

$D(0,0) = -144 < 0 \Rightarrow (0,0,0)$ is a saddle point

$$\left.\begin{aligned} D\left(-\frac{36^{3/5}}{3}, 36^{1/5}\right) &\approx 720 > 0 \\ z_{xx}\left(-\frac{36^{3/5}}{3}, 36^{1/5}\right) &\approx -17.1716 < 0 \end{aligned}\right\} \Rightarrow \approx (-2.9, 2.0, 29.3) \text{ is a local maximum point}$$

6. $z = f(x,y) = x^3 - xy + \frac{y^2}{2}$

$$z_x = 3x^2 - y$$
$$z_y = -x + y$$

$$\begin{matrix} 3x^2 - y = 0 \\ -x + y = 0 \end{matrix} \Rightarrow \begin{matrix} 3x^2 = y \\ x = y \end{matrix} \Rightarrow 3x^2 = x \Rightarrow 3x^2 - x = 0 \Rightarrow x(3x-1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{3}$$

$$\Rightarrow \text{ critical point } = (0,0), \left(\frac{1}{3}, \frac{1}{3}\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -1 \\ -1 & 1 \end{vmatrix} = 6x - 1$$

$D(0,0) = -1 < 0 \Rightarrow (0,0,0)$ is a saddle point

$$\left.\begin{matrix} D\left(\frac{1}{3}, \frac{1}{3}\right) = 1 > 0 \\ z_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = 2 > 0 \end{matrix}\right\} \Rightarrow \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{54}\right) \text{ is a local minimum point}$$

7. $z = f(x,y) = \frac{1}{x} + \frac{1}{y} + xy$

$$z_x = -\frac{1}{x^2} + y$$
$$z_y = -\frac{1}{y^2} + x$$

$$\left.\begin{array}{l} -\frac{1}{x^2} + y = 0 \\ -\frac{1}{y^2} + x = 0 \end{array}\right. \Rightarrow \begin{array}{l} y = \frac{1}{x^2} \\ -\frac{1}{\left(\frac{1}{x^2}\right)^2} + x = 0 \end{array} \Rightarrow -x^4 + x = 0 \Rightarrow x(-x^3 + 1) = 0$$

$\Rightarrow x = 0$ (not possible in the original equation) or $x = 1$

$\Rightarrow$ critical point $= (1,1)$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{vmatrix} = \frac{4}{x^3 y^3} - 1$$

$$\left.\begin{array}{l} D(1,1) = 3 > 0 \\ z_{xx}(1,1) = 2 > 0 \end{array}\right\} \Rightarrow (1,1,3) \text{ is a local minimum point .}$$

8. For a rectangular box of volume 1000 cubic feet, find the dimensions that will minimize the surface area. (Assume the box has a top, bottom, front side, back side, right side, and left side.)

Let *x* = length, *y* = width, and *z* = height. Then $xyz = 1000 \Rightarrow z = \frac{1000}{xy}$, and

$$\text{Surface Area } = S = 2xy + 2yz + 2xz = 2xy + 2y \cdot \frac{1000}{xy} + 2x \cdot \frac{1000}{xy} = 2xy + \frac{2000}{x} + \frac{2000}{y}.$$

$$S_x = 2y - \frac{2000}{x^2}$$
$$S_y = 2x - \frac{2000}{y^2}$$

$$\begin{matrix} 2y - \frac{2000}{x^2} = 0 \\ 2x - \frac{2000}{y^2} = 0 \end{matrix} \Rightarrow \begin{matrix} y - \frac{1000}{x^2} = 0 \\ x - \frac{1000}{y^2} = 0 \end{matrix} \Rightarrow \begin{matrix} y = \frac{1000}{x^2} \\ x - \frac{1000}{\left(\frac{1000}{x^2}\right)^2} = 0 \end{matrix} \Rightarrow x - \frac{x^4}{1000} = 0 \Rightarrow x\left(1 - \frac{x^3}{1000}\right) = 0$$

$\Rightarrow x = 0$ (not possible in this situation) or $x = 10 \Rightarrow$ critical point $= (10,10)$.

$$D = \begin{vmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{vmatrix} = \begin{vmatrix} \frac{4000}{x^3} & 2 \\ 2 & \frac{4000}{y^3} \end{vmatrix}.$$

$$D(10,10) = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 12 > 0 \text{ and } S_{xx}(10,10) = 4 > 0 \Rightarrow \text{ local minimum }.$$

Therefore, the surface area is minimized when *x* = 10 ft, *y* = 10 ft, and *z* = 10 ft.

9. Suppose that $y = mx + b$ is the equation for the line that best fits the points $(0,0),(1,3),$ & $(2,1)$. This means that the sum of the squares $\left(y(0)-0\right)^2 + \left(y(1)-3\right)^2 + \left(y(2)-1\right)^2$ is minimized. Find the equation for the line of best fit.

Let $z = z(m,b) = \left(y(0)-0\right)^2 + \left(y(1)-3\right)^2 + \left(y(2)-1\right)^2 = (b-0)^2 + (m+b-3)^2 + (2m+b-1)^2$. Then,

$$z_m = 2(m+b-3) + 2(2m+b-1)\cdot 2 = 10m + 6b - 10$$
$$z_b = 2b + 2(m+b-3) + 2(2m+b-1) = 6m + 6b - 8$$

$$\begin{matrix} 10m+6b-10=0 \\ 6m+6b-8=0 \end{matrix} \Rightarrow \begin{matrix} 10m+6b=10 \\ 6m+6b=8 \end{matrix} \Rightarrow \begin{matrix} m=\dfrac{1}{2} \\ b=\dfrac{5}{6} \end{matrix} \Rightarrow \text{ criticalt point } = \left(\frac{1}{2},\frac{5}{6}\right)$$

$$D = \begin{vmatrix} z_{mm} & z_{mb} \\ z_{bm} & z_{bb} \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 6 & 6 \end{vmatrix} = 60 - 36 = 24 > 0$$

$$\left.\begin{matrix} D\left(\dfrac{1}{2},\dfrac{5}{6}\right) = 24 > 0 \\ z_{mm}\left(\dfrac{1}{2},\dfrac{5}{6}\right) = 10 > 0 \end{matrix}\right\} \Rightarrow \text{local minimum}$$

Therefore, the line of best fit is $y = \frac{1}{2}x + \frac{5}{6}$.

10. Find three positive numbers whose sum is 48 and whose product is as large as possible, and find that product.

$$x + y + z = 48 \Rightarrow z = 48 - x - y$$

$$P = xyz = xy(48 - x - y) = 48xy - x^2y - xy^2$$

$$P_x = 48y - 2xy - y^2$$
$$P_y = 48x - x^2 - 2xy$$

$$\begin{matrix} 48y - 2xy - y^2 = 0 \\ 48x - x^2 - 2xy = 0 \end{matrix} \Rightarrow \begin{matrix} 48y - 2xy - y^2 = 0 \\ 48x - x^2 = 2xy \end{matrix} \Rightarrow y = \frac{48x - x^2}{2x} = \frac{48 - x}{2}$$

$$\Rightarrow 48\left(\frac{48-x}{2}\right) - 2x\left(\frac{48-x}{2}\right) - \left(\frac{48-x}{2}\right)^2 = 0 \Rightarrow \frac{3}{4}x^2 - 48x + 576 = 0 \Rightarrow x = 48 \text{ or } x = 16.$$

Since $x = 48$ results in $y = z = 0$ and $P = 0$, the only solution of interest here is $x = 16$. Thus, the critical point is $(16,16)$.

$$D = \begin{vmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{vmatrix} = \begin{vmatrix} -2y & 48 - 2x - 2y \\ 48 - 2x - 2y & -2x \end{vmatrix}$$

$$\left.\begin{matrix} D(16,16) = \begin{vmatrix} -32 & -16 \\ -16 & -32 \end{vmatrix} = 1024 - 256 = 768 > 0 \\ P_{xx}(16,16) = -32 < 0 \end{matrix}\right\} \Rightarrow \text{local maximum at } (16,16,16).$$

and $P = 16 \cdot 16 \cdot 16 = 4,096$

11. Use the 2nd partials test to find the point in the plane $2x + y - z = -5$ that is closest to the origin. (HINT: Minimize the square of the distance from the origin. You will get the same answer, but you won't have to mess with derivatives of square roots.)

Note that $z = 2x + y + 5$, and let

$$f = x^2 + y^2 + z^2 = x^2 + y^2 + (2x + y + 5)^2 = 5x^2 + 2y^2 + 4xy + 20x + 10y + 25.$$

$$f_x = 10x + 4y + 20$$
$$f_y = 4x + 4y + 10$$

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \Rightarrow \begin{matrix} 10x + 4y + 20 = 0 \\ 4x + 4y + 10 \end{matrix} \Rightarrow \begin{matrix} x = -5/3 \\ y = -5/6 \end{matrix}$$

Thus, the critical point is $(-5/3, -5/6)$.

$$D(-5/3, -5/6) = \begin{vmatrix} f_{xx}(-5/3, -5/6) & f_{xy}(-5/3, -5/6) \\ f_{yx}(-5/3, -5/6) & f_{yy}(-5/3, -5/6) \end{vmatrix} = \begin{vmatrix} 10 & 4 \\ 4 & 4 \end{vmatrix} = 40 - 16 = 24 > 0$$

$$f_{xx}(-5/3, -5/6) = 10 > 0 \Rightarrow \text{ local minimum}.$$

Therefore, the closest point to the origin is $\left(-\frac{5}{3}, -\frac{5}{6}, \frac{5}{6}\right)$.

12. A company operates two plants which manufacture the same item. Suppose that the cost of operating each plant as a function of the quantities produced is $C_1 = 4q_1^2 + 10$ and $C_2 = q_2^2 + 5$. Suppose also that the total cost is $C = C_1 + C_2$, the total product demand is $q = q_1 + q_2$, and the product price as a function of demand is $p = 90 - q$. Find levels of production, q_1 and q_2, that will maximize the profit.

Revenue is $R = qp = (q_1 + q_2)(90 - q_1 - q_2) = -q_1^2 - q_2^2 - 2q_1q_2 + 90q_1 + 90q_2$.

Profit is $P = R - C = -5q_1^2 - 2q_2^2 - 2q_1q_2 + 90q_1 + 90q_2 - 15$.

$$P_{q_1} = -10q_1 - 2q_2 + 90$$
$$P_{q_2} = -2q_1 - 4q_2 + 90$$

$$\begin{matrix} P_{q_1} = 0 \\ P_{q_2} = 0 \end{matrix} \Rightarrow \begin{matrix} -10q_1 - 2q_2 + 90 = 0 \\ -2q_1 - 4q_2 + 90 = 0 \end{matrix} \Rightarrow \begin{matrix} q_1 = 5 \\ q_2 = 20 \end{matrix}$$

$$D(5,20) = \begin{vmatrix} P_{q_1q_1} & P_{q_1q_2} \\ P_{q_2q_1} & P_{q_2q_2} \end{vmatrix} = \begin{vmatrix} -10 & -2 \\ -2 & -4 \end{vmatrix} = 40 - 4 = 36 > 0$$

$P_{q_1q_1}(5,20) = -10 \Rightarrow$ local maximum.

Therefore, the profit is maximized when $q_1 = 5$ and $q_2 = 20$

13. What numbers *x* and *y* come closest to satisfying the three equations $x - y = 1$, $2x + y = -1$, and $x + 2y = 1$? Solve by minimizing the sum of the squared error terms, $x - y - 1$, $2x + y + 1$, and $x + 2y - 1$.

Let
$$\begin{aligned} z &= (x-y-1)^2 + (2x+y+1)^2 + (x+2y-1)^2 \\ &= x^2 + 2xy + y^2 - 2x + 2y + 1 + 4x^2 + 4x + y^2 + 4xy + 2y + 1 \\ &\quad + x^2 - 2x + 4y^2 + 4xy - 4y + 1 \\ &= 6x^2 + 6xy + 6y^2 + 3. \end{aligned}$$

Then $z_x = 12x + 6y$ and $z_y = 6x + 12y$. Hence,

$$\begin{matrix} z_x = 0 \\ \underline{z_y = 0} \end{matrix} \Rightarrow \begin{matrix} 12x + 6y = 0 \\ \underline{6x + 12y = 0} \end{matrix} \Rightarrow \begin{matrix} 2x + y = 0 \\ \underline{x + 2y = 0} \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}.$$

Also, $\begin{matrix} z_{xx} = 12 & z_{xy} = 6 \\ z_{yx} = 6 & z_{yy} = 12 \end{matrix} \Rightarrow D(0,0) = 144 - 36 = 108 > 0$.

Furthermore, since $z_{xx}(0,0) = 12 > 0$, it now follows that our function is minimized when $x = 0$ and $y = 0$.