SPHERICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in spherical coordinates.

1. Let V be a sphere with center at the origin and radius $= r$. Find the volume of V.

$$\iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\varphi \Big|_0^r d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}\sin\varphi \, d\varphi d\theta = \int_0^{2\pi} -\frac{r^3}{3}\cos\varphi \Big|_0^{\pi} d\theta = \int_0^{2\pi} -\frac{r^3}{3}(\cos\pi - \cos 0)\, d\theta$$

$$= \int_0^{2\pi} \frac{2r^3}{3} d\theta = \frac{2r^3}{3}\Big|_0^{2\pi} = \frac{4}{3}\pi r^3$$

2. Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{x^2 + y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{1} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\sin\varphi \Big|_0^1 d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\varphi \Big|_0^{\frac{\pi}{4}} d\theta = \int_0^{2\pi}\left(-\frac{1}{3\sqrt{2}} + \frac{1}{3}\right) d\theta = \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right)\theta \Big|_0^{2\pi}$$

$$= \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right) 2\pi$$

3. Evaluate $\iiint_V \frac{1}{x^2+y^2+z^2}\,dV$ where V is the solid region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$$\iiint_V \frac{1}{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \frac{1}{\rho^2}\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \rho\sin\varphi \Big|_2^3 \, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi} -\cos\varphi \Big|_0^{\pi} \, d\theta = \int_0^{2\pi} 2\, d\theta$$

$$= 2\theta\Big|_0^{2\pi} = 4\pi$$

4. Find the volume of the solid bounded above by $z=1$ and below by $z=\sqrt{x^2+y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\sin\varphi \Big|_0^{\sec\varphi} \, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\sec^3\varphi}{3}\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\sin\varphi}{3\cos^3\varphi}\, d\varphi\, d\theta = \int_0^{2\pi} \frac{1}{6\cos^2\varphi}\Big|_0^{\frac{\pi}{4}} \, d\theta = \int_0^{2\pi} \left(\frac{1}{3}-\frac{1}{6}\right) d\theta = \int_0^{2\pi} \frac{1}{6}\, d\theta$$

$$= \frac{\theta}{6}\Big|_0^{2\pi} = \frac{\pi}{3}$$

5. Suppose you drill a hole of radius 1 through the center of a sphere of radius 3. Find the volume of the portion removed by the drill.

$$\text{Volume } = 36\pi - 2\int_0^{2\pi}\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}}\int_{\csc\varphi}^{3} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = 36\pi - 2\int_0^{2\pi}\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left.\frac{\rho^3}{3}\sin\varphi\right|_{\csc\varphi}^{3} d\varphi d\theta$$

$$= 36\pi - 2\int_0^{2\pi}\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left(9\sin\varphi - \frac{1}{3}\csc^2\varphi\right) d\varphi d\theta = 36\pi - 2\int_0^{2\pi} \left.\left(-9\cos\varphi + \frac{1}{3}\cot\varphi\right)\right|_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} d\theta$$

$$= 36\pi - 2\int_0^{2\pi}\left(-\frac{1}{3}\cot\left(\sin^{-1}\left(\frac{1}{3}\right)\right) + 9\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)\right) d\theta$$

$$= 36\pi - 2\int_0^{2\pi}\left(\frac{-2\sqrt{2}}{3} + \frac{18\sqrt{2}}{3}\right) d\theta = 36\pi - 2\int_0^{2\pi} \frac{16\sqrt{2}}{3} d\theta = 36\pi - 2\cdot \left.\frac{16\sqrt{2}}{3}\theta\right|_0^{2\pi}$$

$$= 36\pi - \frac{64\sqrt{2}\pi}{3} = 36\pi - \frac{2^{11/2}\cdot 2\pi}{3} = \left(18 - \frac{2^{11/2}}{3}\right)\cdot 2\pi$$

6. Find the volume of the solid region defined by $\rho = \sin\varphi$ where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$.

$$\text{Volume } = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^3}{3}\sin\varphi\right|_0^{\sin\varphi} d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} \sin^4\varphi \, d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} (\sin^2\varphi)^2 \, d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1-\cos 2\varphi}{2}\right)^2 d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\frac{1-2\cos 2\varphi + \cos^2 2\varphi}{4} d\varphi d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\int_0^{\pi} 1 - 2\cos 2\varphi + \frac{1+\cos 4\varphi}{2} d\varphi d\theta = \frac{1}{12}\int_0^{2\pi}\int_0^{\pi}\left(\frac{3}{2} - 2\cos 2\varphi + \frac{\cos 4\varphi}{2}\right) d\varphi d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} \left.\left(\frac{3\varphi}{2} - \sin 2\varphi + \frac{\sin 4\varphi}{8}\right)\right|_0^{\pi} d\theta = \frac{\pi}{8}\int_0^{2\pi} d\theta = \left.\frac{\pi\theta}{8}\right|_0^{2\pi} = \frac{\pi^2}{4}$$

7. An object occupies the region inside the unit sphere with center at the origin and has density equal to the square of the distance from the origin. Find the mass.

$$\text{Mass } = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \cdot \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4 \sin\varphi \, d\rho d\varphi d\theta \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^5}{5}\sin\varphi\right|_0^1 d\varphi d\theta = \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \left.\left(-\cos\varphi\right)\right|_0^{\pi} d\theta = \frac{1}{5}\int_0^{2\pi} 2\, d\theta = \left.\frac{2}{5}\theta\right|_0^{2\pi} = \frac{4\pi}{5}$$