

SURFACE INTEGRALS - ANSWERS

In each problem below you are given a surface S , defined by $z = f(x, y)$, over a region R , defined by the given limits on x and y . You are also given a function $g = g(x, y, z)$. Use

the relation $dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = \sqrt{z_x^2 + z_y^2 + 1} dA$ to find the value of the surface

integral of g on the surface S by evaluating

$$\begin{aligned}\text{Surface Integral} &= \iint_S g(x, y, z) dS = \iint_R g(x, y, z) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA \\ &= \iint_R g(x, y, z) \sqrt{z_x^2 + z_y^2 + 1} dA.\end{aligned}$$

$$S : z = y$$

1. $R : 0 \leq x \leq 1, 0 \leq y \leq 2$

$$g(x, y, z) = x + y + z$$

$$g(x, y, z) = x + y + z = x + 2y$$

$$dS = \sqrt{z_x^2 + z_y^2 + 1} dA = \sqrt{2} dA$$

$$\begin{aligned}\iint_S g dS &= \iint_R (x + 2y) \cdot \sqrt{2} dA = \sqrt{2} \int_0^1 \int_0^2 (x + 2y) dy dx \\ &= \sqrt{2} \int_0^1 xy + y^2 \Big|_0^2 dx = \sqrt{2} \int_0^1 (2x + 4) dx = \sqrt{2} \cdot (x^2 + 4x) \Big|_0^1 = 5\sqrt{2}\end{aligned}$$

$$S: z = x^2 + y^2 + 1$$

2. $R: 0 \leq x \leq 1, 0 \leq y \leq 1$

$$g(x, y, z) = \frac{z}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$g(x, y, z) = \frac{z}{\sqrt{4x^2 + 4y^2 + 1}} = \frac{x^2 + y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$dS = \sqrt{z_x^2 + z_y^2 + 1} dA = \sqrt{4x^2 + 4y^2 + 1} dA$$

$$\begin{aligned} \iint_S g dS &= \iint_R \frac{x^2 + y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}} \cdot \sqrt{4x^2 + 4y^2 + 1} dA = \int_0^1 \int_0^1 (x^2 + y^2 + 1) dy dx \\ &= \int_0^1 \left(x^2 y + \frac{y^3}{3} + y \right) \Big|_0^1 dx = \int_0^1 \left(x^2 + \frac{4}{3} \right) dx = \left(\frac{x^3}{3} + \frac{4}{3}x \right) \Big|_0^1 = \frac{5}{3} \end{aligned}$$

$$S: z = -x - y + 2$$

3. $R: 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}$

$$g(x, y, z) = \cos x + \sin y$$

$$g(x, y, z) = \cos x + \sin y$$

$$dS = \sqrt{z_x^2 + z_y^2 + 1} dA = \sqrt{3} dA$$

$$\begin{aligned} \iint_S g dS &= \iint_R (\cos x + \sin y) \cdot \sqrt{3} dA = \sqrt{3} \int_0^{\pi/2} \int_0^{\pi/2} (\cos x + \sin y) dy dx \\ &= \sqrt{3} \int_0^{\pi/2} \left(y \cos x - \cos y \right) \Big|_0^{\pi/2} dx = \sqrt{3} \int_0^{\pi/2} \left(\frac{\pi}{2} \cos x + 1 \right) dx = \sqrt{3} \left(\frac{\pi}{2} \sin x + x \right) \Big|_0^{\pi/2} \\ &= \sqrt{3} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \pi \sqrt{3} \end{aligned}$$