TRIPLE INTEGRALS - ANSWERS

For each of the following problems write a triple integral and evaluate it.

1. Find the volume of the solid in the first octant bounded above by the plane $x+y+z=1$ and below by the *xy*-plane.

$$\int_0^1\int_0^{-x+1}\int_0^{1-x-y} dzdydx = \int_0^1\int_0^{-x+1}(1-x-y)\,dydx = \int_0^1\left(y-xy-\frac{y^2}{2}\right)\Bigg|_0^{-x+1} dx$$

$$=\int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx = \frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{2}\Bigg|_0^1 = \frac{1}{6}-\frac{1}{2}+\frac{1}{2}=\frac{1}{6}$$

2. Find the volume of the solid between $z=x^2+y^2+1$ and $z=-x^2-y^2-1$ where $-1\le x\le 1$ and $-1\le y\le 1$.

$$\int_{-1}^1\int_{-1}^1\int_{-x^2-y^2-1}^{x^2+y^2+1} dzdydx = \int_{-1}^1\int_{-1}^1\left(x^2+y^2+1\right)-\left(-x^2-y^2-1\right)dydx$$

$$=\int_{-1}^1\int_{-1}^1\left(2x^2+2y^2+2\right)dydx = \int_{-1}^1\left(2x^2y+\frac{2y^3}{3}+2y\right)\Bigg|_{-1}^1 dx$$

$$=\int_{-1}^1\left(4x^2+\frac{4}{3}+4\right)dx = \left(\frac{4x^3}{3}+\frac{4x}{3}+4x\right)\Bigg|_{-1}^1 = \left(\frac{4}{3}+\frac{4}{3}+4\right)-\left(-\frac{4}{3}-\frac{4}{3}-4\right)$$

$$=\frac{8}{3}+\frac{8}{3}+\frac{24}{3}=\frac{40}{3}$$

3. Evaluate $\iiint_V xe^{xy}\,dV$ where V is the solid bounded by the planes $x=-1$, $x=1$, $y=0$, $y=2$, $z=0$, and $z=3$.

$$\int_{-1}^1\int_0^2\int_0^3 xe^{xy}\,dzdydx = \int_{-1}^1\int_0^2 xe^{xy}z\Big|_0^3 dydx = \int_{-1}^1\int_0^2 3xe^{xy}\,dydx = \int_{-1}^1 3e^{xy}\Big|_0^2 dx$$

$$=\int_{-1}^1\left(3e^{2x}-3\right)dx = \left(\frac{3e^{2x}}{2}-3x\right)\Bigg|_{-1}^1 = \left(\frac{3e^2}{2}-3\right)-\left(\frac{3e^{-2}}{2}+3\right)=\frac{3(e^2-e^{-2})}{2}-6$$

4. Evaluate $\int_5^6\int_3^4\int_1^2 \cos(x)\cos(y)\cos(z)\,dzdydx$.

$$\int_5^6\int_3^4\int_1^2 \cos x\cos y\cos z\,dzdydx = \int_5^6\int_3^4 \cos x\cos y\sin z\Big|_1^2\,dydx$$
$$= \int_5^6\int_3^4 \cos x\cos y\left(\sin 2-\sin 1\right)dydx = \int_5^6 \cos x\sin y(\sin 2-\sin 1)\Big|_3^4\,dx$$
$$= \int_5^6 \cos x(\sin 4-\sin 3)(\sin 2-\sin 1)\,dx = \sin x(\sin 4-\sin 3)(\sin 2-\sin 1)\Big|_5^6$$
$$= (\sin 6-\sin 5)(\sin 4-\sin 3)(\sin 2-\sin 1)$$

5. Evaluate $\int_0^1\int_0^z\int_0^y ze^{y^2}\,dxdydz$

$$\int_0^1\int_0^z\int_0^y ze^{y^2}\,dxdydz = \int_0^1\int_0^z xze^{y^2}\Big|_0^y\,dydz = \int_0^1\int_0^z yze^{y^2}\,dydz$$
$$= \int_0^1 \frac{ze^{y^2}}{2}\Bigg|_0^z dz = \int_0^1\left(\frac{ze^{z^2}}{2}-\frac{z}{2}\right)dz = \frac{e^{z^2}}{4}-\frac{z^2}{4}\Bigg|_0^1 = \left(\frac{e}{4}-\frac{1}{4}\right)-\left(\frac{1}{4}-0\right)$$
$$= \frac{e-2}{4}.$$

6. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos\theta\,dzdrd\theta$

$$\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos\theta\,dzdrd\theta = \int_0^{\pi/2}\int_0^{\sin\theta} rz\cos\theta\Big|_0^{r\sin\theta}\,drd\theta = \int_0^{\pi/2}\int_0^{\sin\theta} r^2\sin\theta\cos\theta\,drd\theta$$
$$= \int_0^{\pi/2} \frac{r^3}{3}\sin\theta\cos\theta\,d\theta\Bigg|_0^{\sin\theta}\,d\theta = \int_0^{\pi/2}\frac{1}{3}\sin^4\theta\cos\theta\,d\theta = \frac{\sin^5\theta}{15}\Bigg|_0^{\pi/2} = \frac{1}{15}.$$

7. Find the volume of the solid region bounded by $z=|x|+|y|$ and $z=2$.

We can take advantage of symmetry by finding the volume in just the first octant and then multiplying the result by four.

$$4\int_0^2\int_0^{-x+2}\int_{x+y}^{2} dzdydx = 4\int_0^2\int_0^{-x+2}(2-x-y)\,dydx = 4\int_0^2\left(2y-xy-\frac{y^2}{2}\right)\Bigg|_0^{-x+2} dx$$
$$=4\int_0^2\left(\frac{x^2}{2}-2x+2\right)dx=\int_0^2\left(2x^2-8x+8\right)dx=\left(\frac{2x^3}{3}-4x^2+8x\right)\Bigg|_0^2$$
$$=\frac{16}{3}-16+16=\frac{16}{3}$$

8. Use a triple integral to find the volume of the great pyramid of Cheops given that it is 482 feet tall and has a square base that is 754 feet on each side. (HINT: Take advantage of symmetry.)

$$\text{volume } =8\int_0^{377}\int_0^{x}\int_0^{-\frac{482}{377}x+482} dzdydx=8\int_0^{377}\int_0^{x}\left(-\frac{482}{377}x+482\right)dydx$$
$$=8\int_0^{377}\left(-\frac{482}{377}x+482\right)y\Bigg|_0^{x} dx=8\int_0^{377}\left(-\frac{482}{377}x^2+482x\right)dx$$
$$=8\left(-\frac{482}{377}\frac{x^3}{3}+241x^2\right)\Bigg|_0^{377}=8\left(-\frac{482\cdot 377^2}{3}+241\cdot 377^2\right)$$
$$=91{,}341{,}570.67\ ft^3$$

9. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one of the corners.

$$\text{density } =\int_0^2\int_0^2\int_0^2 x^2+y^2+z^2\,dzdydx=\int_0^2\int_0^2 zx^2+zy^2+\frac{z^3}{3}\Bigg|_0^2 dydx$$
$$=\int_0^2\int_0^2 2x^2+2y^2+\frac{8}{3}dydx=\int_0^2 2yx^2+\frac{2y^3}{3}+\frac{8}{3}y\Bigg|_0^2 dx=\int_0^2 4x^2+\frac{32}{3}dx$$
$$=\frac{4x^3}{3}+\frac{32x}{3}\Bigg|_0^2=\frac{32}{3}+\frac{64}{3}=\frac{96}{3}=32.$$

10. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one of the edges.

$$\text{density } = \int_0^2\int_0^2\int_0^2 y^2 + z^2 \, dzdydx = \int_0^2\int_0^2 zy^2 + \frac{z^3}{3}\bigg|_0^2 \, dydx$$

$$= \int_0^2\int_0^2 2y^2 + \frac{8}{3} dydx = \int_0^2 \frac{2y^3}{3} + \frac{8}{3}y\bigg|_0^2 \, dx = \int_0^2 \frac{32}{3} dx$$

$$= \frac{32x}{3}\bigg|_0^2 = \frac{64}{3}.$$