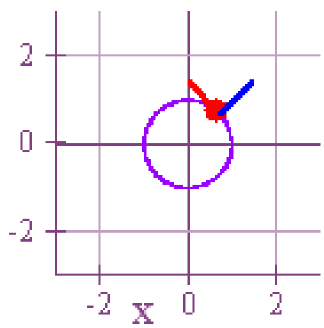
UNIT TANGENTS AND NORMALS - ANSWERS

(1-5) For each of the following curves, find the unit tangent and the unit normal at the indicated value for *t*. Also, graph the curve along with the unit tangent and normal you found.

1. $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$, $0 \le t \le 2\pi$, $t = \dfrac{\pi}{4}$

$$\vec{r}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$
$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$
$$T\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$
$$N\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$
$$\vec{r}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\hat{i} + \sin\left(\frac{\pi}{4}\right)\hat{j} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$



2. $\vec{r}(t) = 2\cos(t)\hat{i} - 2\sin(t)\hat{j}$, $0 \le t \le 2\pi$, $t = \dfrac{5\pi}{4}$

$$\vec{r}'(t) = -2\sin(t)\hat{i} - 2\cos(t)\hat{j}$$

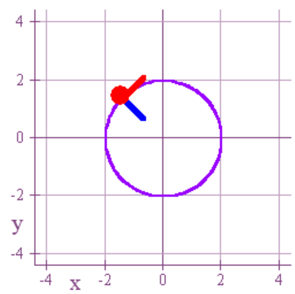
$$\|\vec{r}'(t)\| = \sqrt{(-2\sin t)^2 + (-2\cos t)^2} = 2$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} - \cos(t)\hat{j}$$

$$T\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$N\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$$\vec{r}\left(\frac{5\pi}{4}\right) = 2\cos\left(\frac{5\pi}{4}\right)\hat{i} - 2\sin\left(\frac{5\pi}{4}\right)\hat{j} = -\sqrt{2}\hat{i} + \sqrt{2}\hat{j}$$



3. $\vec{r}(t) = (2+3t)\hat{i} + (1+4t)\hat{j}$, $0 \le t \le 2$, $t = 1$

$$\vec{r}'(t) = 3\hat{i} + 4\hat{j}$$

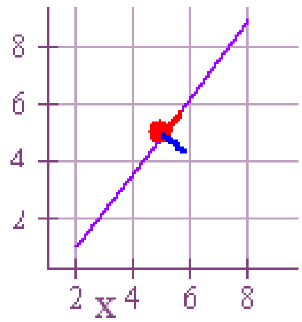
$$\|\vec{r}'(t)\| = \sqrt{3^2 + 4^2} = 5$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$T(1) = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$N(1) = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}$$

$$\vec{r}(1) = 5\hat{i} + 5\hat{j}$$



4. $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}\,,\ -2 \le t \le 2\,,\ t = 1$

$$\vec{r}\,'(t) = \hat{i} + (2t)\,\hat{j}$$

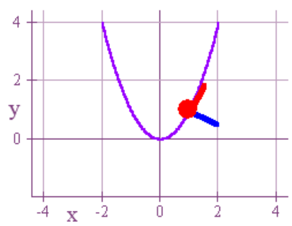
$$\left\|\vec{r}\,'(t)\right\| = \sqrt{1+4t^2}$$

$$T(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} = \frac{1}{\sqrt{1+4t^2}}\,\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\,\hat{j}$$

$$T(1) = \frac{1}{\sqrt{5}}\,\hat{i} + \frac{2}{\sqrt{5}}\,\hat{j}$$

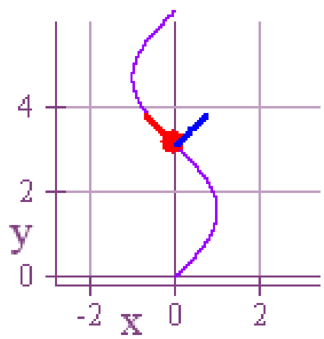
$$N(1) = \frac{2}{\sqrt{5}}\,\hat{i} - \frac{1}{\sqrt{5}}\,\hat{j}$$

$$\vec{r}(1) = \hat{i} + \hat{j}$$



5. $\vec{r}(t)=\sin t\hat{i}+t\hat{j}$, $0\le t\le 2\pi$, $t=\pi$

$$\vec{r}'(t)=\cos(t)\hat{i}+\hat{j}$$
$$\|\vec{r}'(t)\|=\sqrt{1+\cos^2 t}$$
$$\|\vec{r}'(\pi)\|=\sqrt{1+\cos^2 \pi}=\sqrt{2}$$
$$T(t)=\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}=\frac{\cos t}{\sqrt{1+\cos^2 t}}\hat{i}+\frac{1}{\sqrt{1+\cos^2 t}}\hat{j}$$
$$T(\pi)=-\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$$
$$N(\pi)=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$$
$$\vec{r}(\pi)=\sin(\pi)\hat{i}+\pi\hat{j}=\pi\hat{j}$$



6. If $T(t)$ is the unit tangent vector for a curve describe by $\vec{r}(t)$, then show that T and T' are perpendicular to one another.

Clearly, for any t we have that $\|T(t)\|=1$, a constant. Therefore, by pervious proof, $T\bullet T'=0$ and T is perpendicular to T'.