$E = mc^2$ in One Page!

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Two things need to be mentioned at the very start. First, according to Einstein's Theory of Special Relativity, an object's mass increases as its velocity increases. The relationship between its moving mass m and its rest mass m_0 is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}.$$

Consequently, we have the following derivative.

$$\frac{dm}{dv} = m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\left(-\frac{2v}{c^2}\right) = \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0 v}{\left(c^2-v^2\right)\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{v}{\left(c^2-v^2\right)} = m\cdot\frac{v}{c^2-v^2} \Rightarrow m = \frac{c^2-v^2}{v}\cdot\frac{dm}{dv}$$

The next item to be clear about is the formula from Newtonian physics that $Force = mass \times acceleration$.

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In differential equations, since acceleration is the rate of change of velocity with respect to time, we usually write this as

$$F = ma = m\frac{dv}{dt} = \frac{d(mv)}{dt}.$$

In Newtonian physics the mass is constant, but in Einstein's world it isn't. Thus, if we expand $F = \frac{d(mv)}{dt}$ using the product rule, then we get

$$F = \frac{d(mv)}{dt} = m\cdot\frac{dv}{dt} + v\cdot\frac{dm}{dt}.$$

Making a substitution from above for m, we get

$$F = \frac{d(mv)}{dt} = m\cdot\frac{dv}{dt} + v\cdot\frac{dm}{dt} = \frac{c^2-v^2}{v}\cdot\frac{dm}{dv}\cdot\frac{dv}{dt} + v\cdot\frac{dm}{dt} = \frac{c^2-v^2}{v}\cdot\frac{dm}{dt} + v\cdot\frac{dm}{dt}.$$

In differential form, this looks like

$$F\,dt = \frac{c^2-v^2}{v}dm + v\,dm$$

If we now let x represent position so that $v = \frac{dx}{dt}$, then we can do the following algebraic magic with our differentials.

$$F\,dx = F\,dt\cdot\frac{dx}{dt} = \left[\frac{c^2-v^2}{v}dm + v\,dm\right]\cdot v \Rightarrow F\,dx = \left(c^2-v^2\right)dm + v^2\,dm = c^2\,dm - v^2\,dm + v^2\,dm = c^2\,dm$$

The bottom line is now,

$$F\,dx = c^2\,dm.$$

However, $F\,dx$ is simply *Force* × *change in distance* = *change in work*. Furthermore, in classical physics *change in work* = *change in kinetic energy*. Thus, our formula now becomes

$$dE = c^2 dm \Rightarrow \int dE = \int c^2 dm \Rightarrow E = mc^2$$ (initial condition: $E=0$ when $m=0$).

And it's just that simple!

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