Essential Theorems for Harmonic Astrology

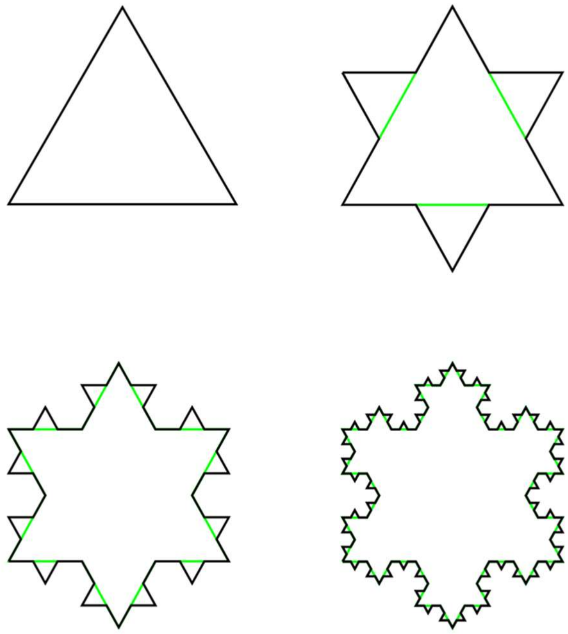
Christopher P. Benton, PhD

Abstract

Harmonic astrology began its development by astrologer John Addey during the late 1950s, and the variant known as vibrational astrology was developed by Santino Mancuso (aka David Cochrane) several decades later, but the roots of both in Vedic astrology have been around for hundreds of years. During this time, many rules for computation and interpretation have been developed, but the one thing that has frequently been lacking is a set of rigorous mathematical proofs of the essentials of this highly mathematical branch of astrology. This is unfortunate as good mathematical models are often essential for a discipline to advance. Hence, in this work proofs are presented for several essential theorems that apply to harmonic astrology, and by doing so greater clarity can be brought to the underlying structure of this branch of astrology. Furthermore, this paper is also designed to be two papers in one. In other words, those who are comfortable with mathematical rigor can read the theorems and proofs in full detail while those who are not used to such rigor can read just the examples and other surrounding content to understand the results contained herein. Furthermore, a link to videos that provide additional explanation of harmonics and the proofs below can be found in the references at the end of this article (Benton, 2024).

Introduction: Fractals

One of the earliest examples of a fractal in mathematics (even though the concept was not formally identified until approximately 70 years later) is the Koch snowflake. This is a snowflake image that is generated by starting with an equilateral triangle and repeatedly replacing line segments by the same image over and over again. Below is an illustration of how this begins and what the end result resembles. You start with an equilateral triangle, and you keep replacing straight line segments by a curve that deviates from a straight line by drawing the top portion of a triangle over the middle third. This replacement procedure is then repeated ad infinitum, and the result is shown below in the following illustration from Wikipedia.

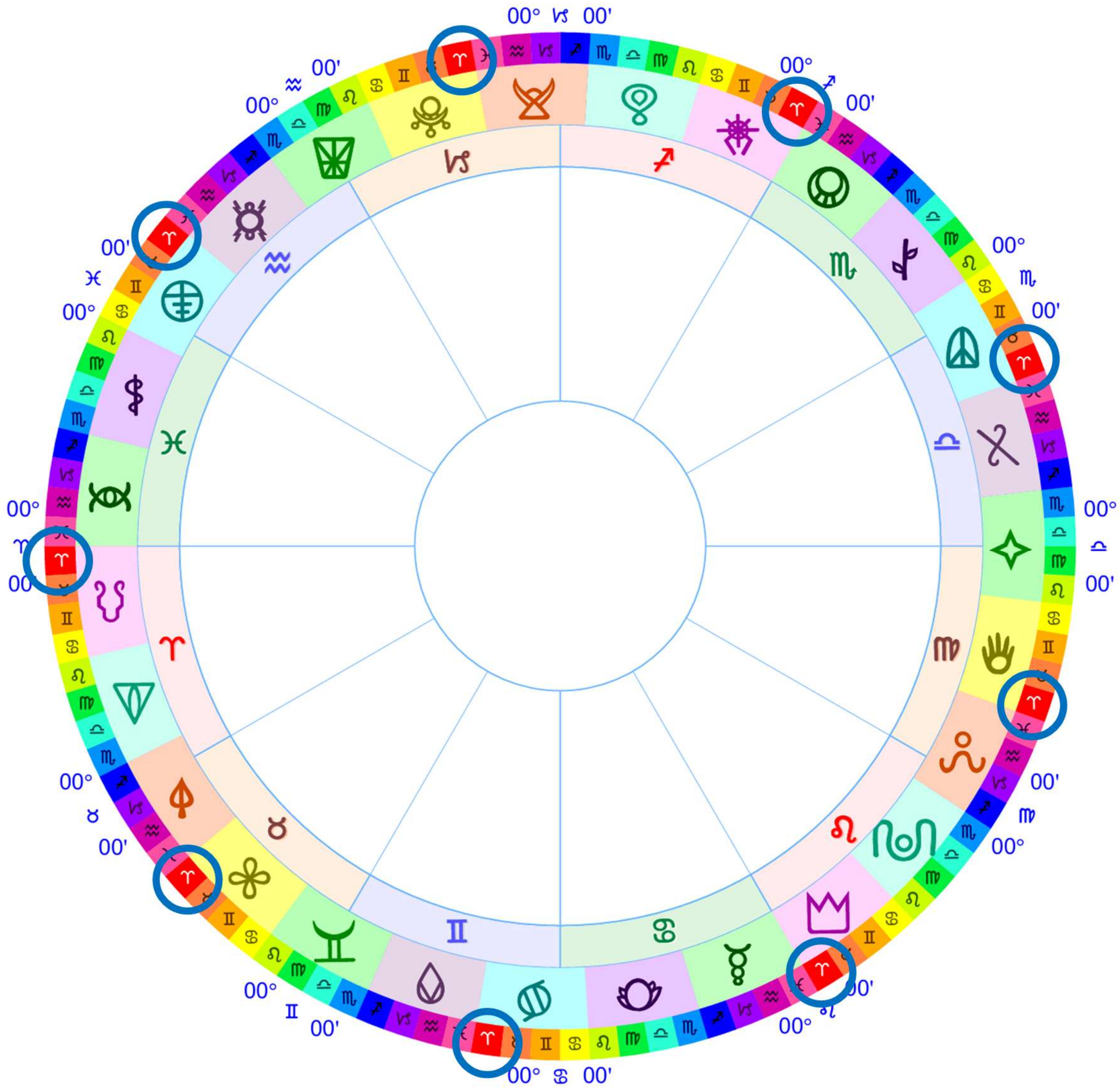


The reason for beginning with the Koch snowflake is because harmonics in astrology are constructed in basically the same way. However, instead of replacing line segments with other curves, angles, such as 90°, will be replaced with entire zodiacs, and if this type of replacement is continued ad infinitum, then an astrological fractal that is a counterpart to the Koch snowflake will be created. In this paper, though, only a single level of sub-zodiacs will be constructed. However, in a planned follow-up paper on the qualitative interpretation of harmonics, several sub-levels of sub-zodiacs will be considered, and that will more closely mirror the process for the construction of the Koch snowflake. Furthermore, in the follow-up to this paper, the act of replacing an original angle by an entire sub-zodiac will add additional nuance to the horoscope interpretation.

As an example of the Vedic origins of harmonic charts, below in Figure1 is a diagram showing the structure of the *navamsa* chart that corresponds to the 9 harmonic chart in harmonic astrology. The inner ring of the chart wheel shows the 12 signs of the traditional Western zodiac. The middle ring, which will not be utilized in this paper, shows the 27 *nakshatras* or lunar mansions as they are depicted in Vedic astrology. The outer ring, however, shows what happens if one subdivides the original zodiac into 9 equal parts of 40° and then replaces each part with an entire zodiac. In other words, the original zodiac is replaced by 9 sub-zodiacs, and each sub-zodiac comprises 40° of the original 360° circle. The result is a geometric representation of the *navamsa* or 9 harmonic, and to find the longitudinal position of any planet in the 9 harmonic, one simply needs to determine its location in the corresponding sub-zodiac.

Figure 1

The subdivision of a zodiac into 9 equal parts leads to what Vedic astrology calls the navamsa chart and what Western astrology calls the 9 harmonic chart. Furthermore, the starting points for each of the 9 sub-zodiacs are indicated below by the blue circles.



As a further example, if a horoscope wheel is divided into two equal halves with the first half going from 0° (Aries) to 180° (Libra) and the second half going from 180° (Libra) to 360° (Aries), and if each half is replaced by an entire zodiac from Aries through Pisces (See Figure 2), then the fundamental question is what longitudinal position will a planet or other point from the original horoscope occupy in one of these sub-zodiacs? The answer to this question can be found through simple proportions. For example, suppose that the angle in the initial horoscope is 120° (0° Leo). That angle is two-thirds of the way through the interval from 0° to 180° and so it must also be two-thirds of the way through the 360° sub-zodiac that is replacing it. This results in the proportion $\frac{120°}{180°} = \frac{x}{360°}$ which implies that $x = 240° = 0°$ Sagittarius (See Figure 3).

Likewise, 300° is two-thirds of the way through the interval from 180° to 360°, and so if this interval is replaced with a 360° sub-zodiac, then 300° will correspond to 240° in the 360° sub-zodiac since, again, this is the point that is two-thirds of the way from $0°$ to $360°$ (See Figure 4). Furthermore, since $180° = \frac{360°}{2}$, $180°$ is called the root 2 harmonic angle, and the astrological chart that is generated by replacing every important angle (either an angle from $0°$ Aries to the chart point, or the angle between two chart points) in the original horoscope by its corresponding angle from one of the two sub-zodiacs is called the 2 harmonic chart. This is illustrated geometrically in Figures 2, 3, and 4.

Figure 2

If a natal chart is divided into 2 equal parts (180°) and each part is replaced by an entire zodiac, then the 2 harmonic is being constructed.

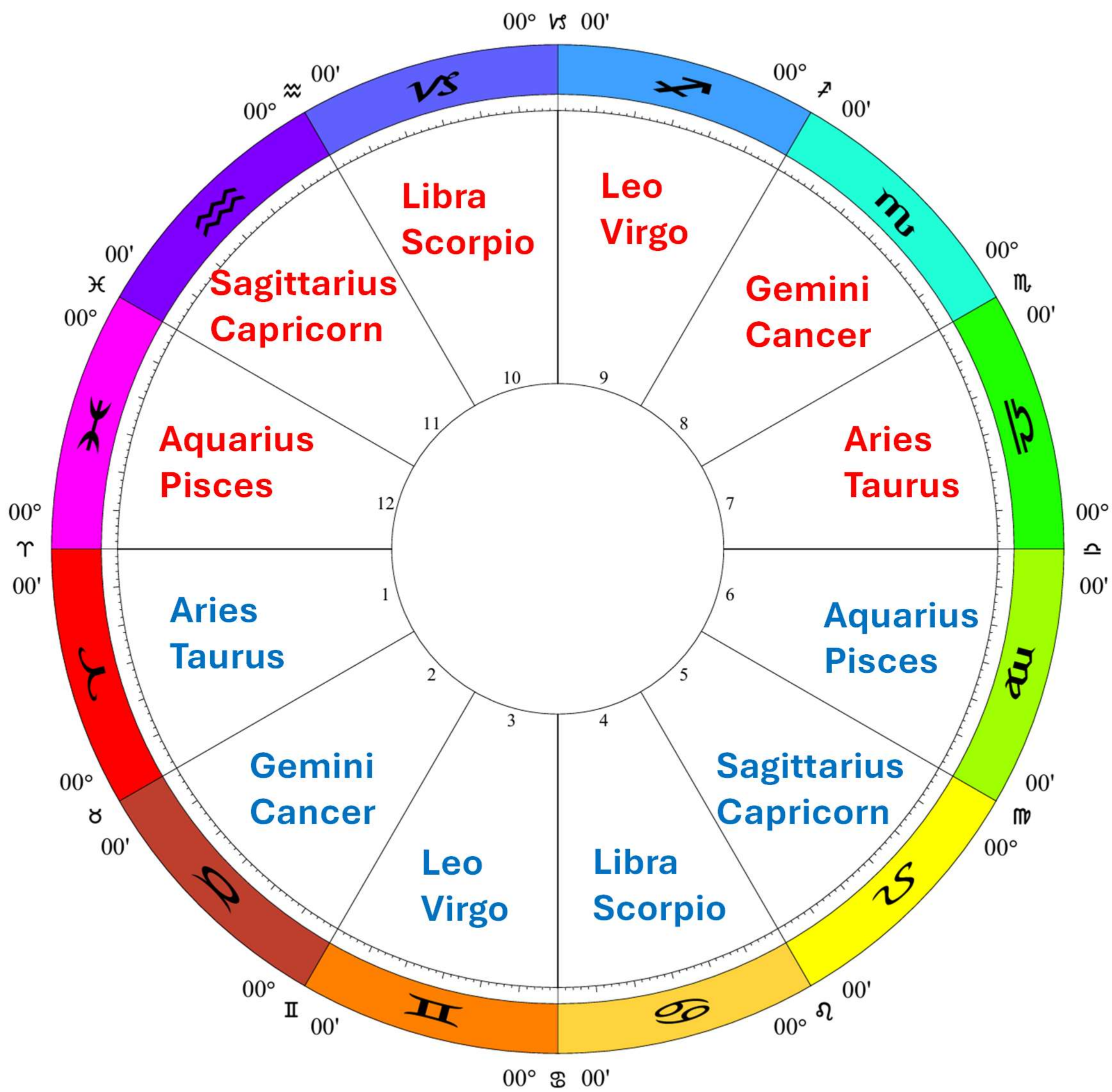
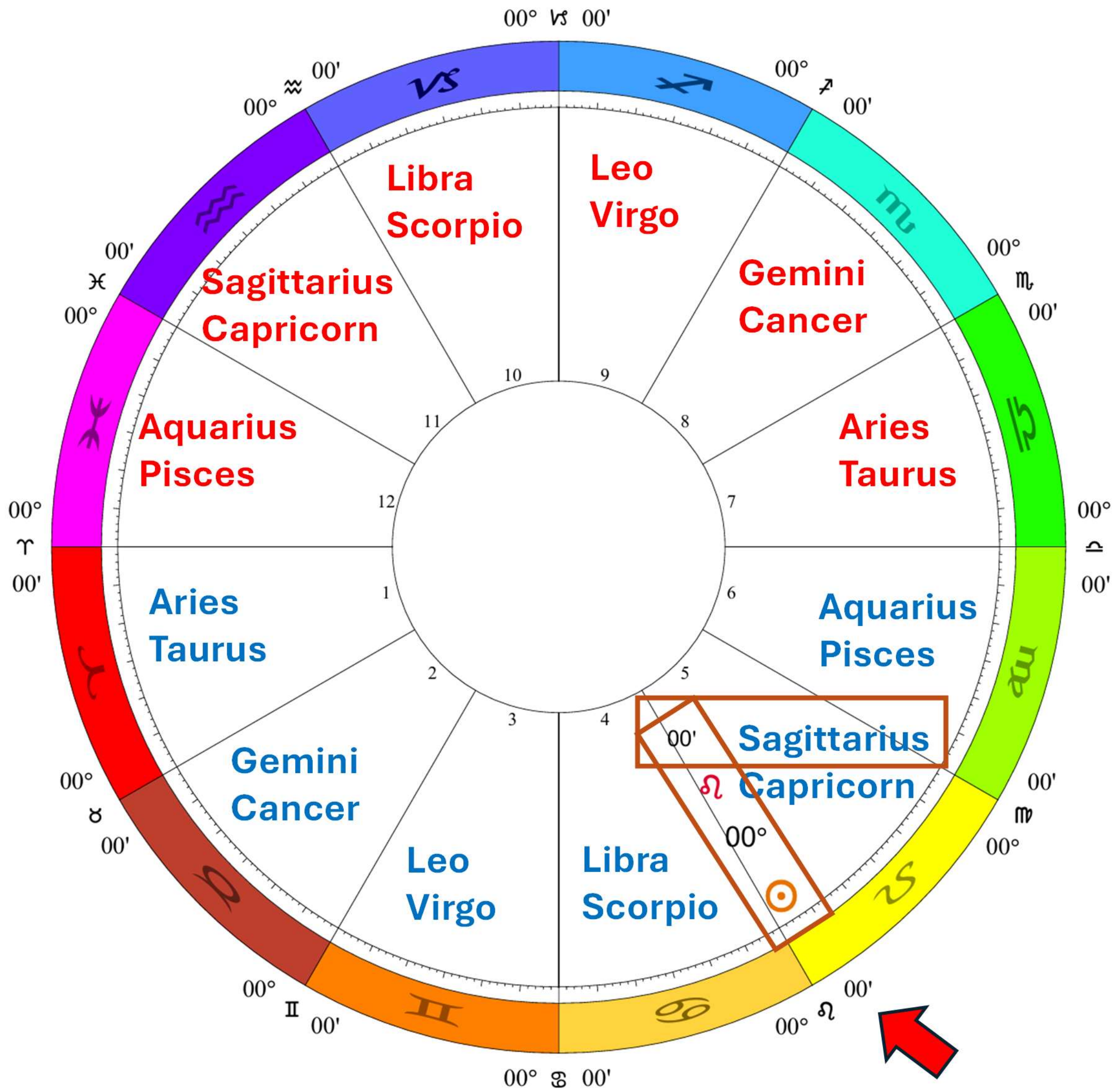


Figure 3

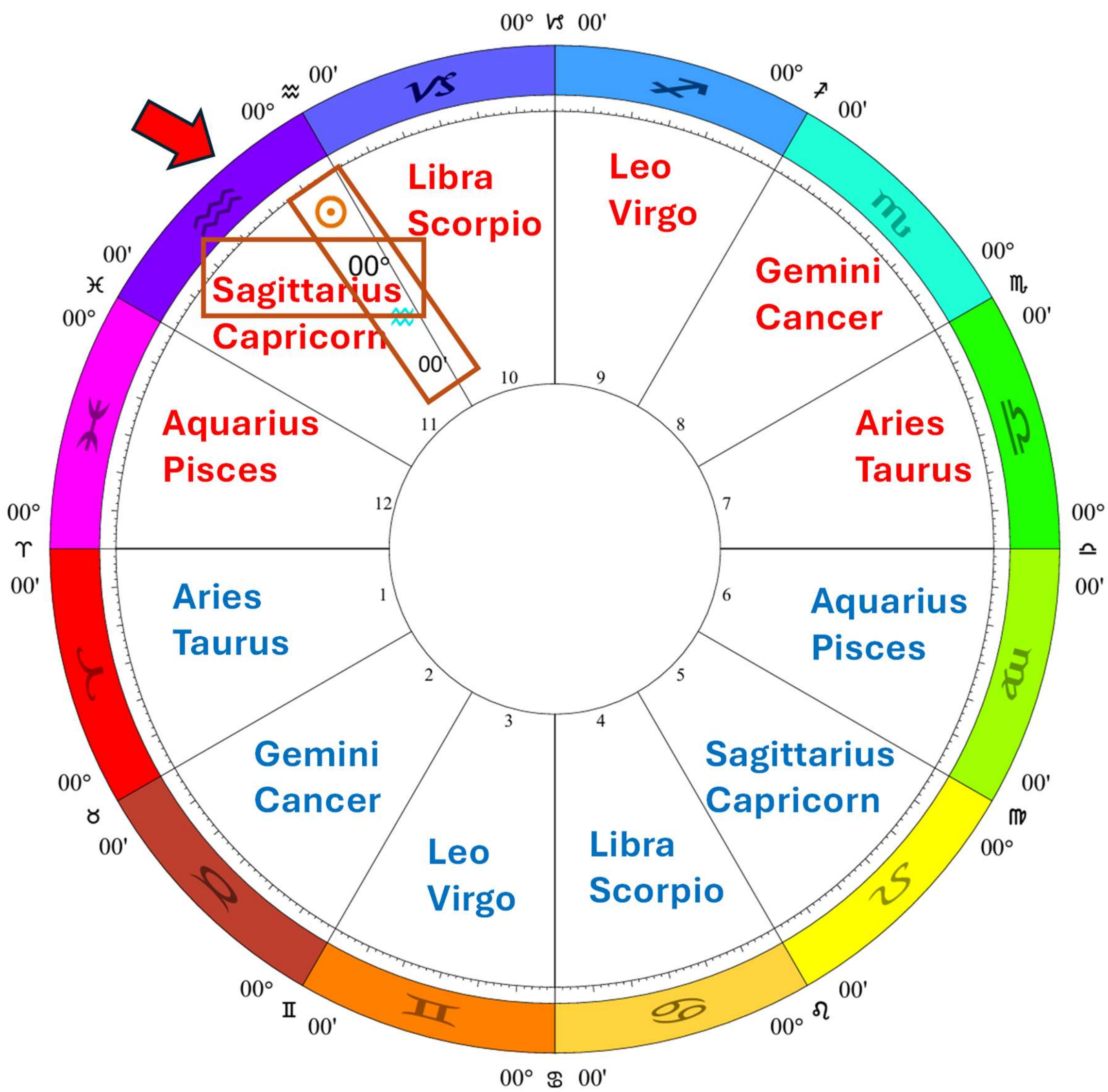
In the 2 harmonic chart, $0°$ *Leo corresponds to* $0°$ *Sagittarius*.



Libra
Scorpio
Leo
Virgo
Sagittarius
Capricorn
Gemini
Cancer
Aquarius
Pisces
Aries
Taurus
Aries
Taurus
Aquarius
Pisces
Gemini
Cancer
Sagittarius
Capricorn
Leo
Virgo
Libra
Scorpio
00'
00°

Figure 4

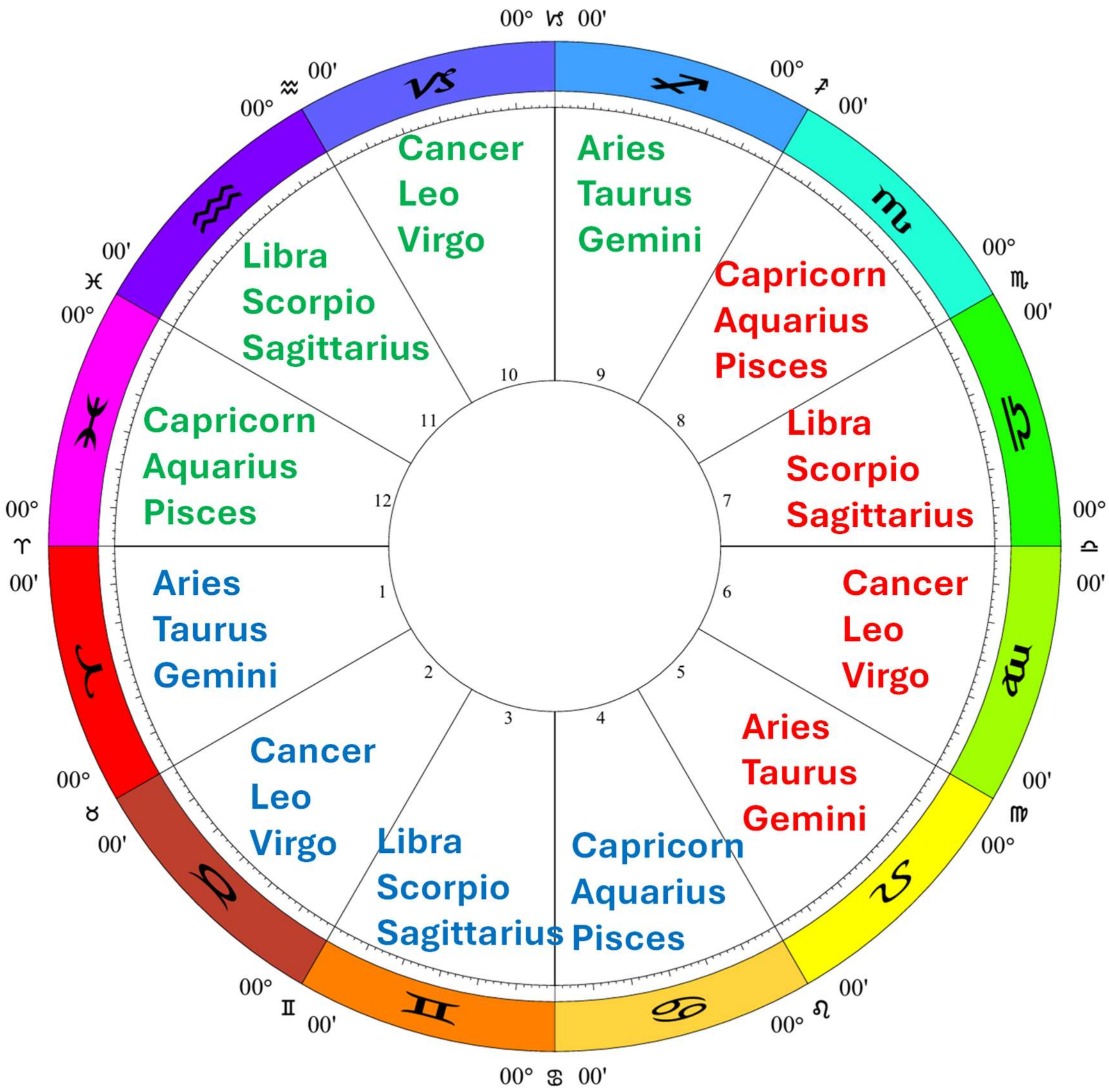
In the 2 harmonic chart, 0° Aquarius corresponds to 0° Sagittarius.



To construct the 3 harmonic, the original zodiac wheel needs to be divided into three sub-zodiacs as is shown in Figure 5.

Figure 5

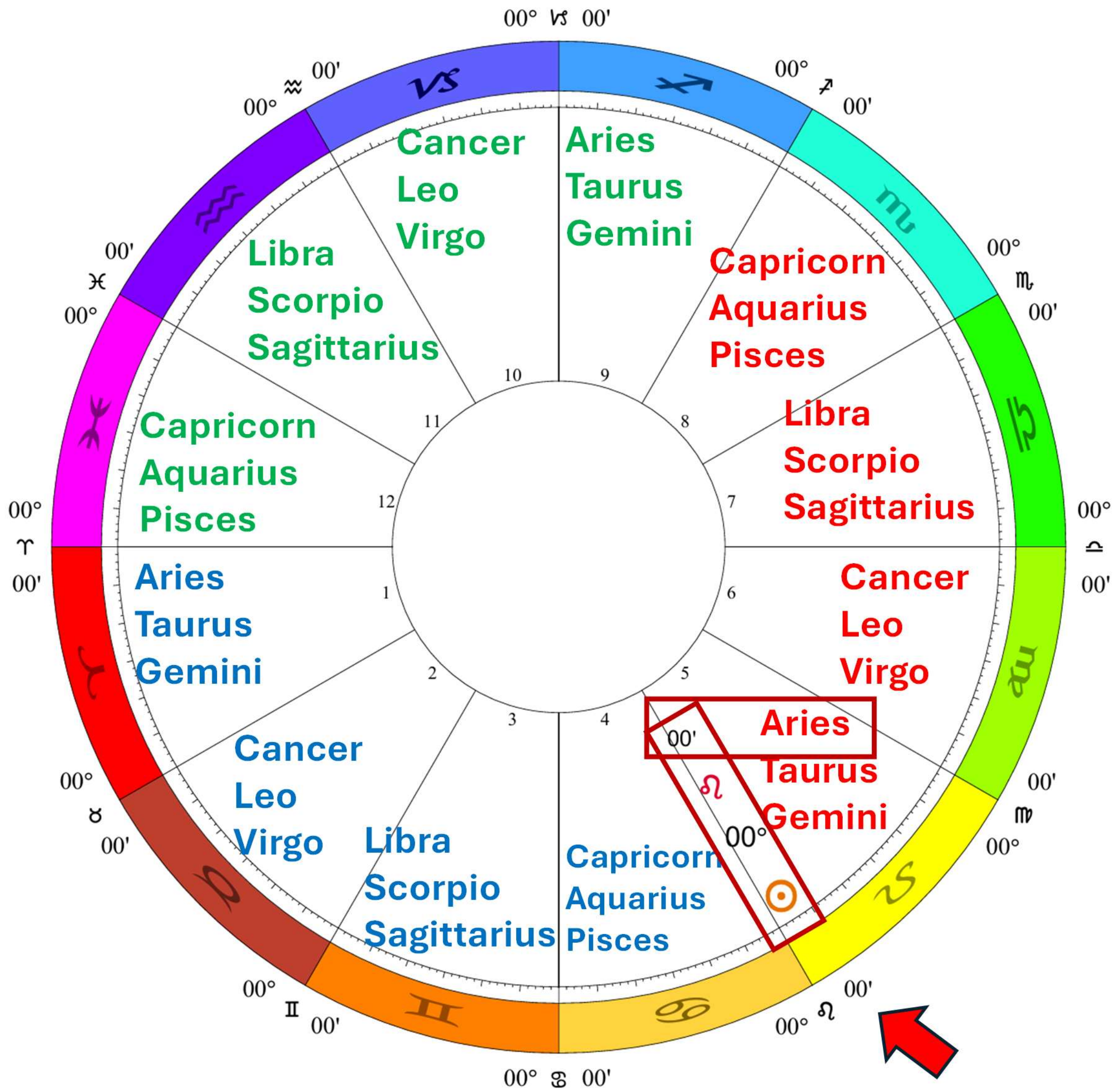
If a natal chart is divided into 3 equal parts (120°) and each part is replaced by an entire zodiac, then the 3 harmonic is being constructed.



Using this geometric representation of the 3 harmonic, one can easily see (figures 6 & 7) that 0° Leo corresponds to 0° Aries, and 0° Aquarius corresponds to 0° Libra.

Figure 6

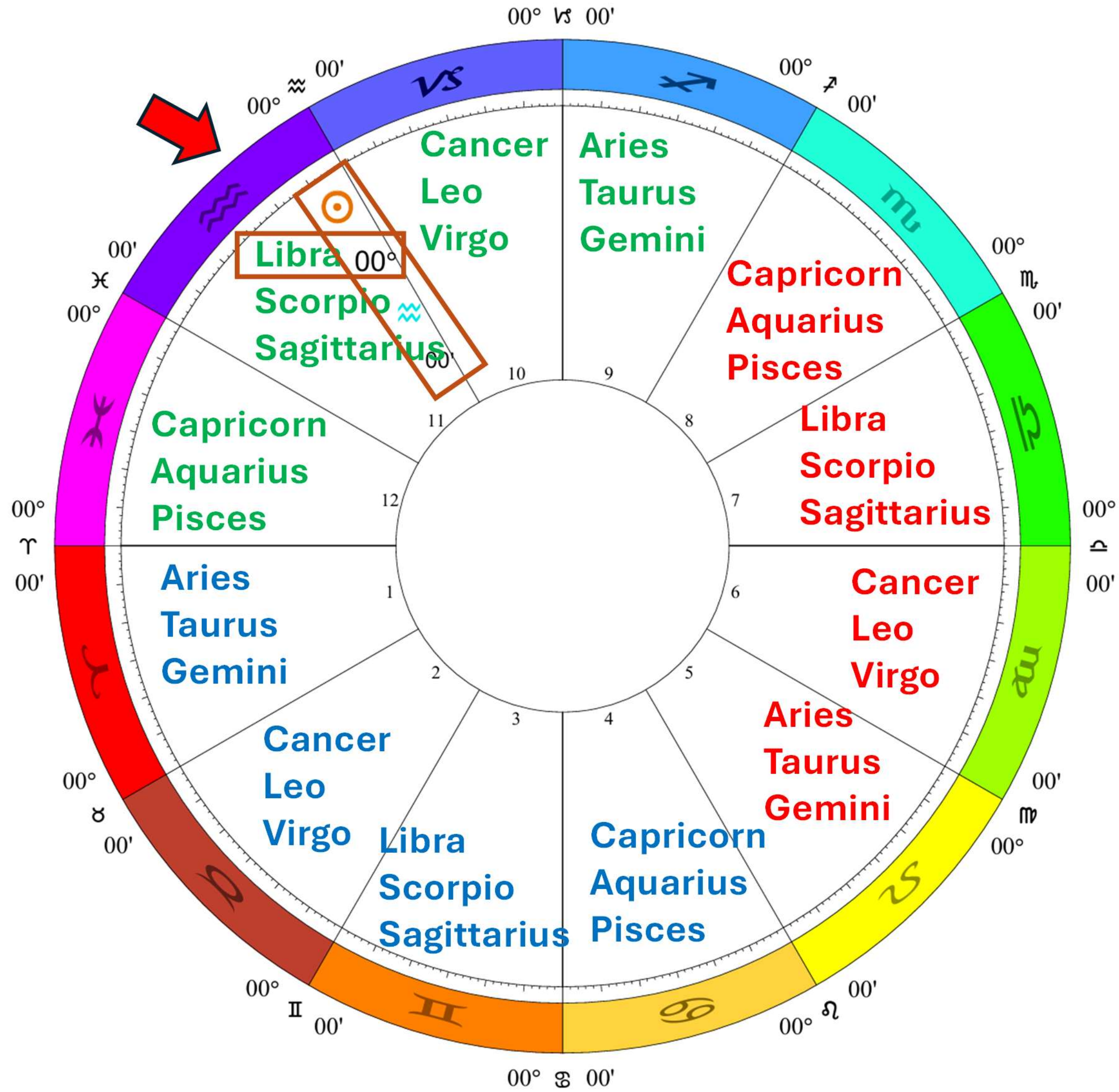
In the 3 harmonic chart, 0° *Leo corresponds to* 0° *Aries*.



Similarly, as seen in Figure 7, 0° Aquarius corresponds to 0° Libra in the 3 harmonic.

Figure 7

In the 3 harmonic chart, 0° *Aquarius corresponds to* 0° *Libra*.



In this presentation, theorems and results will be shown only for counting number (natural number) harmonics and positive rational number (fractional) harmonics, and extensive use will be made of modular arithmetic. This is similar to what many people know as "clock arithmetic" where once one reaches 12, you literally start the clock over with 1 as the next hour. Thus, for example, instead of saying "16 o'clock," one either subtracts 12 or looks at the remainder after division by 12, and in each case, one gets 4 o'clock instead of 16 o'clock. Also, in mathematics one always starts the clock with 0 instead of 12 since 12 divided by 12 yields a remainder of 0. However, this paper will not restrict itself to doing clock arithmetic just *modulo* 12, to use the mathematical term for clock arithmetic, and most of the time the focus will be on results $modulo\ 360°$ since that is the common way to divide up a circle in both astrology and mathematics. Hence, results like $240°\ modulo\ 360° = 240°$, $420°\ modulo\ 360° = 60°$, and $720°\ modulo\ 360° = 0°$ will be commonly seen.

Harmonics in astrology can also be defined for negative numbers and for irrational numbers, and many of the theorems below will also apply to these other numbers, but they are not generally needed. Negative numbers can be ignored for now for the same reason that one can always say 9 o'clock instead of negative 3 o'clock when denoting the time, and irrational numbers, while theoretically important, do not obey many of the rules that apply to other harmonics and in practice they can always be closely approximated with any desired degree of accuracy by a rational number. With that said and in order to keep things simple (for now!), the primarily focus will be on counting number and positive rational number harmonics.

Additionally, because of the difficulty that many will encounter in trying to understand mathematical proofs, this paper is designed to be read on two levels. In particular, if one simply skips all of the theorems and their proofs, then what is left will still be a comprehensive introduction to the mathematical structure of harmonic astrology that is designed to be accessible to a much larger audience.

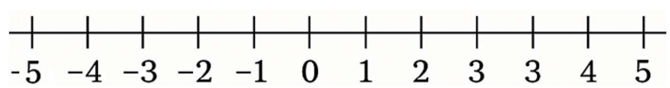
Now, before continuing, a few definitions will be made.

Definitions

Definition 1: A **counting or natural number** is any one of the numbers $1, 2\ 3\ 4, \dots$ that a person usually counts with. The set of counting numbers is symbolized by $\mathbb{N}$, and these numbers are also called natural numbers because people often think of things in nature as occurring in these units. For example, one rock, two rocks, and so on.

Definition 2: A **whole number** is any one of the numbers $0,1,2,3,4, \dots$. The set of whole numbers is sometimes symbolized by a block letter $\mathbb{W}$.

Definition 3: An **integer** is any one of the numbers $0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$. The set of integers is symbolized by $\mathbb{Z}$, and the set of positive integers is symbolized by $\mathbb{Z}^+$. Also, the integers are the numbers that are usually marked off for the scale on a number line as shown below.



Definition 4: A **rational number** is a number that can be written as a ratio of two integers such as $\frac{1}{2}$ or $\frac{2}{3}$ or $\frac{-6}{5}$. A **rational number** can also be defined as a number whose decimal expansion is either terminating such as 0.125 or is non-terminating and repeating (periodic) such as $5.232323 \dots$. The set of rational numbers is symbolized by $\mathbb{Q}$, and the set of positive rational numbers is symbolized by $\mathbb{Q}^+$. Furthermore, if $\frac{n}{m}$ is a rational number and if *n* and *m* have no common counting number divisor (see Definition 9) other than 1, then $\frac{n}{m}$ is said to be in **lowest terms**.

Definition 5: The **irrational numbers** are numbers with decimal expansions that are both non-repeating and non-terminating such as $\sqrt{2}$ or π. One can also create an irrational number by simply constructing a number with an infinite decimal expansion that lacks a fixed sequence that repeats such as, for example, $0.101001000100001 \ldots$. Since one always adds in this construction an additional "0" between any two successive 1s as one goes from left to right, this guarantees that there will be no fixed sequence of digits that repeats, and so the number is irrational. The set of these numbers is sometimes symbolized by $\mathbb{H}$ with the **positive irrational numbers** represented by $\mathbb{H}^+$.

Definition 6: The **real numbers** are the numbers that correspond to points on the number line. They also consist of the rational numbers combined with the irrational numbers, and they are symbolized by $\mathbb{R}$. Similarly, the **positive real numbers** are those real numbers that are greater than zero, and they can be symbolized by $\mathbb{R}^+$.

Definition 7: The ***least common multiple*** or ***lcm*** of two counting numbers is the smallest counting number that is a counting number multiple of both[1]. Additionally, the *least common multiple* can also be found for fractions. In particular, if $a, b, c,$ and d are counting numbers, then $lcm\left(\frac{a}{b}, \frac{c}{d}\right) = lcm\left(\frac{ad}{bd}, \frac{bc}{bd}\right) = \frac{lcm\,(ad,bc)}{bd}$.

For the numbers 4 and 6, it is not difficult to see that 12 is the *least common multiple* (*lcm*) of these two numbers. In this case, $12 = 3 \cdot 4 = 2 \cdot 6$. But what if one wants to find the *lcm* of two rational numbers (fractions) like $\frac{4}{3}$ and $\frac{6}{5}$? The definition above says that one should first rewrite each fraction with a common denominator, $\frac{20}{15}$ and $\frac{18}{15}$. Next, one finds the *lcm* of the two numerators, and then one can end by reducing the resulting fraction to lowest terms. In other words, $lcm\left(\frac{4}{3}, \frac{6}{5}\right) = lcm\left(\frac{20}{15}, \frac{18}{15}\right) = \frac{lcm(20,18)}{15} = \frac{180}{15} = 12$. Notice, too, that since $12 = 18 \cdot \frac{2}{3}$ and $12 = 10 \cdot \frac{6}{5}$, it is a counting number multiple of each fraction. A computational shortcut, though, for finding the *least common multiple* of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is to divide the *least common multiple* of the numerators by the *greatest common divisor* (see Definition 8) of the denominators, i.e. $lcm\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{lcm(a,c)}{gcd\,(b,d)}$. For example, using this shortcut formula, $lcm\left(\frac{4}{3}, \frac{6}{5}\right) = \frac{lcm(4,6)}{gcd\,(3,5)} = \frac{12}{1} = 12$.

Definition 8: The ***greatest common divisor*** or ***gcd*** of two counting numbers is the largest counting number that yields a counting number when divided into both[2]. Additionally, the *greatest common divisor* can also be found for fractions. In particular, if $a, b, c,$ and d are counting numbers, then $gcd\left(\frac{a}{b}, \frac{c}{d}\right) = gcd\left(\frac{ad}{bd}, \frac{bc}{bd}\right) = \frac{gcd\,(ad,bc)}{bd}$.

For the numbers 4 and 6, it is not difficult to see that 2 is the *greatest common divisor* (*gcd*) of the two numbers. In this case, $\frac{4}{2} = 2$ and $\frac{6}{2} = 3$, and the numbers 2 & 3 have no factor left in common other than 1. But if one wants to find the *greatest common divisor* of $\frac{4}{3}$ and $\frac{6}{5}$, then the definition says that $gcd\left(\frac{4}{3}, \frac{6}{5}\right) = gcd\left(\frac{20}{15}, \frac{18}{15}\right) = \frac{gcd\,(20,18)}{15} = \frac{2}{15}$. Again, however, there is a shortcut formula, $gcd\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{gcd\,(a,c)}{lcm(b,d)}$, that can be used for this computation. Thus, using the shortcut, $gcd\left(\frac{4}{3}, \frac{6}{5}\right) = \frac{gcd\,(4,6)}{lcm(3,5)} = \frac{2}{15}$.

Definition 9: If a and b are real numbers, then a is a **divisor** of b if and only if there exists an integer k such that $k = b/a$ with no non-zero remainder. [Note that in higher abstract algebra, a divisor is often defined in terms of multiplication rather than the division $b/a = k$. This is so that one can get around $0/0$ being undefined and still claim that 0 is a divisor of 0. For this paper, though, the more common definition in terms of division will suffice.]

Definition 10: If r is a positive rational number, then the angle $\frac{360°}{r}$ is called the **root *r* angle** and $\theta_r = \left(\frac{360°}{r}\right)$ *modulo* $360°$ is called the **root r harmonic angle**. Additionally, **r** is called the corresponding **harmonic**. Note that if $r > 1$, then the root *r* angle will be the same as the root *r* harmonic angle.

Definition 11: If θ_r is a root r harmonic angle for some positive rational number *r*, then a $\boldsymbol{r}$ **harmonic angle** is any angle of the form $(k \cdot \theta_r)\ modulo\ 360°$ where k is a counting number. Notice that for a counting number $n > 1$ and root *n* harmonic angle $\theta_n = \frac{360°}{n}$, the number of distinct *n* harmonic angles is equal to *n,* and they are $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n\theta_n)\ modulo\ 360° = 0°$. This last fact will be proven later as Theorem 4.

Definition 12: If *r* is a positive rational number and $\theta_r = \left(\frac{360°}{r}\right)\ modulo\ 360°$ is the root r harmonic angle, then the set of all *r* harmonic angles will be denoted by $\boldsymbol{H_r = \{(k \cdot \theta_r)\ modulo\ 360°\}}$ where *k* is a counting number. Furthermore, H_r will be considered to be an ordered set where the order of the elements corresponds to increasing values of *k*, and this will be called the **natural order** for H_r. For example, $H_3 = \{120°, 240°, 0°\}$ where the angles are given in their natural order.

A few more advanced definitions will be added when the need arises, but before that, some methods for calculating harmonic charts will be derived.

Calculating Harmonics

Before going any further, some rules for quickly determining the *r* harmonic for any angle in the zodiac need to be developed. Prior to proving Theorem 1 below, however, a simple example will be given that illuminates the method of proof that will be followed. Thus, suppose that one wants to convert 245° (measured from 0° Aries) to it's value in the 2 harmonic (see Figure 1). Then, as has been seen above, one first needs to determine how far into the appropriate sub-zodiac 245° has gone. Since each sub-zodiac comprises 180° of the original zodiac, this can be done by computing $245°\ modulo\ 180° = 65°$. This shows that 245° has traversed 65° into the second sub-zodiac that, in the 2 harmonic, replaces the 180° interval from Libra through Pisces. To find the value of 65° in this sub-zodiac, a simple proportion can be set up:

$$\frac{65°}{180°} = \frac{n}{360°} \Longrightarrow n = 360° \cdot \frac{65°}{180°} = 2 \cdot 65° = 130°$$

In a similar manner, if one wants to find the 3 harmonic of 245°, then since $\frac{360°}{3} = 120°$, one can first find $245°\ modulo\ 120° = 5°$, and then just solve another simple proportion:

$$\frac{5°}{120°} = \frac{n}{360°} \Longrightarrow n = 360° \cdot \frac{5°}{120°} = 3 \cdot 5° = 15°$$

As yet another example, to find the 4 harmonic of 245°, first calculate $\frac{360°}{4} = 90°$, and find $245°\ modulo\ 90° = 65°$. From here, one can solve what should now be a familiar proportion:

$$\frac{65°}{90°} = \frac{n}{360°} \Longrightarrow n = 360° \cdot \frac{65°}{90°} = 4 \cdot 65° = 260°$$

The proof of Theorem 1 that follows simply mimics, in a more abstract manner, the proportions that were just solved. Furthermore, Theorem 1 suggests the following three-step process for finding the value of an angle *x* in a particular harmonic *r*:

1. Divide 360° by the harmonic r, $\left(\frac{360°}{r}\right)$.
2. Find $x \; modulo \; \frac{360°}{r}$.
3. Multiply $x \; modulo \; \frac{360°}{r}$ by the harmonic r.

To illustrate finding the 4 harmonic of 245° using this algorithm, one just needs to complete each step below.

1. $\frac{360°}{4} = 90°$,
2. $245° \; modulo \; 90° = 65°$.
3. $4 \cdot 65° = 260°$.

Theorem 1: Let x be an angle with $0° \leq x < 360°$, let r be a positive rational number, and suppose one wants to find the value x_r of x in the r harmonic. Then the root r harmonic angle $\frac{360°}{r}$ is the angle measure that will be replaced by a sub-zodiac, and $x_r = r \cdot \left(x \; modulo \; \frac{360°}{r}\right)$.

Proof: Clearly

$$\frac{x \; modulo \; \frac{360°}{r}}{\frac{360°}{r}} = \frac{x_r}{360°}$$

$$\Rightarrow 360° \cdot \left(\frac{x \; modulo \; \frac{360°}{r}}{\frac{360°}{r}}\right) = x_r$$

$$\Rightarrow r \cdot \left(x \; modulo \; \frac{360°}{r}\right) = x_r$$

■

A common application of this theorem is when one is replacing each 30° sign with a 360° zodiac from Aries through Pisces. In other words, when computing the 12 harmonic. To give an example, if one has their Sun is at 2° Gemini, then it is 62° from the beginning of Aries. However, one does not really need to know that in this case because since $30° := \frac{360°}{12}$, the related harmonic is $12 := \frac{360°}{30°}$, and $62° \; modulo \; \frac{360°}{12} = 62° \; modulo \; 30° = 2°$, the degree measure within the sign of Gemini. Hence, the value of 2° Gemini in the 12 harmonic is $12 \cdot 2° = 24° = 24°$ Aries. To give another example, if the Moon is at 27° Taurus, then its position in the 12 harmonic is $12 \cdot 27° = 324° = 24°$ Aquarius.

Again, the formula above is very handy for when one wants to convert a position within a sign to its zodiac position in the 12 harmonic. Nonetheless, most people probably compute all harmonics using the formula shown below in Theorem 2 that says to simply multiply one's angle by the corresponding harmonic and then reduce the result $modulo$ 360°, and this and other formulas apply not only to the angle of a planet's position as measured from 0° Aries but also the angle between any two chart points such as a trine between two planets. As examples of this new procedure, $245°$ will be converted once again to the 2, 3, and 4 harmonics.

1. $(2 \cdot 245°)\ modulo\ 360° = 490°\ modulo\ 360° = 130°$.
2. $(3 \cdot 245°)\ modulo\ 360° = 735°\ modulo\ 360° = 15°$.
3. $(4 \cdot 245°)\ modulo\ 360° = 980°\ modulo\ 360° = 260°$.

Even though the rule derived by Theorem 2 is the common way to compute an angle or position in a harmonic, the rule given by Theorem 1 has the advantage that there is no need to reduce the result $modulo\ 360°$ when done.

Theorem 2: Let x be an angle with $0° \leq x < 360°$, let r be a positive rational number, and suppose one wants to find the angle x_r of x in a r harmonic. Then $x_r = (r \cdot x)\ modulo\ 360°$.

Proof: By Theorem 1, $x_r = r \cdot \left(x\ modulo\ \frac{360°}{r}\right)$. However, notice that one can rewrite the original angle x as $x = w \cdot \frac{360°}{r} + \left(x\ modulo\ \frac{360°}{r}\right)$, where w is a whole number. As an example of this, suppose that $x = 62°$, which corresponds to 2° Gemini, and let $r = 12$ so that $\frac{360°}{12} = 30°$. Then $62°\ modulo\ 30° = 2°$, and one can write, as per our formula, that $62° = 2 \cdot 30° + 2°$. Continuing now with the proof, notice that $(r \cdot x)\ modulo\ 360° = \left\{r \cdot \left[w \cdot \frac{360°}{r} + \left(x\ modulo\ \frac{360°}{r}\right)\right]\right\}\ modulo\ 360° = \left[w \cdot 360° + r\left(x\ modulo\ \frac{360°}{r}\right)\right]\ modulo\ 360° = \left[r\left(x\ modulo\ \frac{360°}{r}\right)\right]\ modulo\ 360°$ since the remainder when one divides $w \cdot 360°$ by $360°$ is 0. Furthermore, notice again that by Theorem 1 that $r\left(x\ modulo\ \frac{360°}{r}\right) = x_r$ where $0° \leq x_r < 360°$, and, thus, it follows that $x_r\ modulo\ 360° = x_r$. Hence,

$(r \cdot x)\ modulo\ 360° =.\left[r\left(x\ modulo\ \frac{360°}{r}\right)\right]\ modulo\ 360° = x_r\ modulo\ 360° = x_r.$

■

As stated before, the formula developed in Theorem 2 is probably the one that most people use when calculating harmonics by hand. For example, instead of using the result of Theorem 1 to find the 12 harmonic of 27° Taurus by calculating $12 \cdot 27° = 324° = 24°$ Aquarius as was done before, one can simply multiply the measure of that angle from 0° Aries (specifically 57°) by 12 and then express the result *modulo* 360°. This yields $(12 \cdot 57°)\ modulo\ 360° = 684°\ modulo\ 360° = 324° = 24°$ Aquarius, the same result obtained previously by using Theorem 1.

Higher Harmonics of Zero Degrees

Before going any further, though, a very simple theorem will be shown that is, nonetheless, also very basic to the structure of harmonic astrology. It essentially tells one that any angle of 0° will remain fixed at 0° in any higher positive rational number harmonic. The proof amounts to simply stating that any number times 0 is still 0. For example, if one wants to compute the 5 harmonic of 0°, then when multiplying 0° by 5, the result is still 0°, $(5 \cdot 0° = 0°)$.

Theorem 3: An angle of 0° will remain at 0° in a r harmonic chart for any positive rational number r.

Proof: Clearly, $r(0°) = 0°$. Hence, $[r(0°)]\ modulo\ 360° = 0°\ modulo\ 360° = 0°$.

■

As an application of theorem 3, suppose that one begins with any New Moon chart where the Sun and the Moon are exactly conjunct. Then any higher harmonic of this chart will also have the Sun and Moon exactly conjunct.

The Number of *n* Harmonics Angles

At first glance, when one is finding a particular harmonic chart, the end result often seems completely random, but nothing could be further from the truth. Instead, there is a very precise mathematical structure to harmonics as the next few theorems will show. The first of these theorems formally proves that for any counting number *n*, there are exactly *n* harmonic angles. Specifically, if one wants to find all 5 harmonic angles, then start with the root 5 harmonic angle, $\frac{360°}{5} = 72°$, and compute the counting number multiples of this angle until one arrives at 360° which is equivalent to 0°. This results in $72°, 2 \cdot 72° = 144°, 3 \cdot 72° = 216°, 4 \cdot 72° = 288°$, and $(5 \cdot 72°)\ modulo\ 360° = 360°\ modulo\ 360° = 0°$. Furthermore, as suggested by Definition 12, it is convenient to write this set of 5 harmonic angles as $H_5 = \{72°. 144°, 216°, 288°, 0°\}$.

Theorem 4: Let *n* be a counting number and let $\theta_n = \left(\frac{360°}{n}\right)\ modulo\ 360°$ be the root *n* harmonic angle. Then there are exactly *n* harmonic angles, and they are equal to $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n\theta_n)\ modulo\ 360° = 0°$.

Proof: Notice that if $n = 1$, then $\theta_1 = \left(\frac{360°}{1}\right)\ modulo\ 360° = 0°$. Now suppose that $n > 1$. Then $\theta_n = \left(\frac{360°}{n}\right)\ modulo\ 360° = \frac{360°}{n}$, and $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n-1)\theta_n$ are all greater than 0° and less than 360°. Additionally, $(n\theta_n)\ modulo\ 360° = \left(n \cdot \frac{360°}{n}\right)\ modulo\ 360° = (360°)\ modulo\ 360° = 0°$. Beyond n, notice that $[(n+1)\theta_n]\ modulo\ 360° = \left[(n+1)\frac{360°}{n}\right]\ modulo\ 360° = \left(360° + \frac{360°}{n}\right) modulo\ 360° = (360° + \theta_n)\ modulo\ 360° = \theta_n\ modulo\ 360° = \theta_n$. Hence, once one reaches $n+1$, one starts over with θ_n, and from this point on one simply repeats the previous values for harmonic angles as one continues multiplying by larger counting numbers. Therefore, for any counting number *n*, the *n* harmonic contains exactly *n* harmonic angles that can be listed as $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n-1)\theta_n, 0°$.

■

As an example of Theorem 4, the root 6 harmonic angle is $60° = \frac{360°}{6}$, and the six 6 harmonic angles are $H_6 = \{60°, 120°, 180°, 240°, 300°, 0°\}$. Similarly, the root 5 harmonic is $72° = \frac{360°}{5}$, and the five 5 harmonic angles are $H_5 = \{72°, 144°, 216°, 288°, 0°\}$.

Next, a perhaps obvious but important corollary to Theorem 4 is presented. It basically says that if you multiply a *n* harmonic angle by any counting number, then even if you have to reduce it $modulo\ 360°$, the end result is still a *n* harmonic angle. For example, if one multiplies the 3 harmonic angle $120°$ by 32, then $(32 \cdot 120°)\ modulo\ 360° = 3840°\ modulo\ 360° = 240°$ which is still a 3 harmonic angle.

Corollary 4.1: If θ is a *n* harmonic angle, then any counting number multiple *k* of θ reduced $modulo\ 360°$ is also one of the *n* harmonic angles $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n-1)\theta_n, 0°$ where $\theta_n = \frac{360°}{n}$.

Proof: Assume that θ is a n harmonic angle Then by Theorem 4, θ is one of the angles $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n-1)\theta_n, 0°$ where $\theta_n = \frac{360°}{n}$. If $\theta = 0°$ and *k* is a counting number, then it is trivial that $(k \cdot \theta)\ modulo\ 360° = 0°\ modulo\ 360° = 0°$ is a *n* harmonic angle. On the other hand, if $\theta \neq 0°$, then $(k \cdot \theta)\ modulo\ 360° = (k \cdot m\theta_n)\ modulo\ 360°$ where m is a counting number such that $1 \leq m \leq n - 1$. But this also means that $(k \cdot \theta)\ modulo\ 360° = (k \cdot m\theta_n)\ modulo\ 360° = (km \cdot \theta_n)\ modulo\ 360°$ is, by definition, a *n* harmonic angle, and so it must also equal one of the angles $\theta_n, 2\theta_n, 3\theta_n, \ldots, (n-1)\theta_n, 0°$ where $\theta_n = \frac{360°}{n}$, and this completes the proof.

■

The practical importance of this corollary is that any counting number multiple of a *n* harmonic angle, when reduced $modulo\ 360°$, is also a *n* harmonic angle.

A Distinguishing Characteristic of *n* Harmonic Angles

This next theorem is also very fundamental in that it tells us that for any counting number $n > 1$, an angle between 0° and 360° is equal to 0° in the *n* harmonic if and only if it is a *n* harmonic angle. For example, the set of 4 harmonic angles is $H_4 = \{90°, 180°, 240°, 0°\}$, and if each angle is multiplied by 4 and reduced $modulo\ 360°$, then in each instance (shown below), one gets back 0° as the end result. Furthermore, if θ is an angle with $0° < \theta < 360°$ and such that $4\theta\ modulo\ 360° = 0°$, then θ is a 4 harmonic angle.

1. $(4 \cdot 90°)\ modulo\ 360° = 360°\ modulo\ 360° = 0°$.
2. $(4 \cdot 180°)\ modulo\ 360° = 720°\ modulo\ 360° = 0°$.
3. $(4 \cdot 270°)\ modulo\ 360° = 1080°\ modulo\ 360° = 0°$.
4. $(4 \cdot 0°)\ modulo\ 360° = 0°\ modulo\ 360° = 0°$.

Theorem 5: For any counting number $n > 1$ and any angle θ with $0° < \theta < 360°$, $(n\theta)\ modulo\ 360° = 0°$ if and only if θ is a *n* harmonic angle.

Proof: ($\Rightarrow$) Suppose that an angle θ with $0° < \theta < 360°$ gets converted to 0° in the n harmonic chart for some counting number $n > 1$. Then $(n\theta)\ modulo\ 360° = 0°$ which means $n\theta = k(360°)$ for some counting number k. But this means that $\theta = k\left(\frac{360°}{n}\right) = k \cdot \theta_n < 360°$, and, hence, θ is a n harmonic angle.

($\Leftarrow$) Suppose that $\theta_n = \left(\frac{360°}{n}\right)\ modulo\ 360°$ is the root n harmonic angle for a counting number $n > 1$. Note that since $\frac{360°}{n} < 360°$, in this case $\theta_n = \left(\frac{360°}{n}\right)\ modulo\ 360° = \frac{360°}{n}$. Also, all other n harmonic angles are counting number multiples of this root harmonic angle calculated $modulo$ $360°$. In particular, if $\theta < 360°$ is a n harmonic angle, then there exists a counting number k such that $\theta = k \cdot \theta_n$. Notice that it is not necessary to reduce this last result $modulo$ $360°$ because it is part of our hypothesis that $\theta < 360°$. Additionally, in the n harmonic chart θ is converted to $[n(\theta)]\ modulo\ 360° = [n(k \cdot \theta_n)]\ modulo\ 360° = [k(n \cdot \theta_n)]\ modulo\ 360° = \left[k\left(n \cdot \frac{360°}{n}\right)\right]\ modulo\ 360° = [k(360°)]\ modulo\ 360° = 0°$. Therefore, if θ is a n harmonic angle for a counting number $n > 1$ with $0° < \theta < 360°$, then θ is converted to $0°$ in the n harmonic chart.

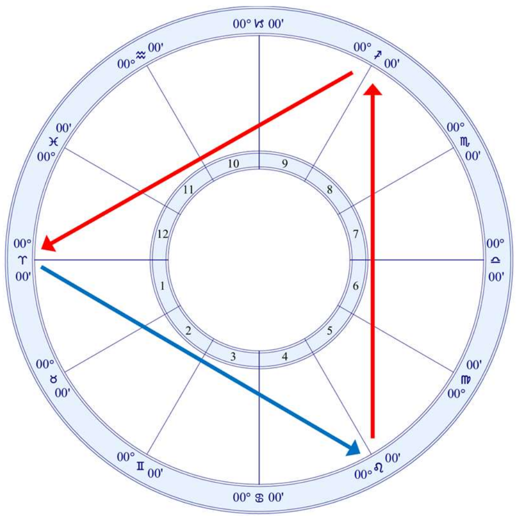
■

One might consider $n = 1$ with $\theta = \frac{360°}{1} = 360°$ as a special case for which the theorem would then be restated as, "$(\theta)\ modulo\ 360° = 0°$ *if and only if* θ *is a whole number multiple of* $360°$."

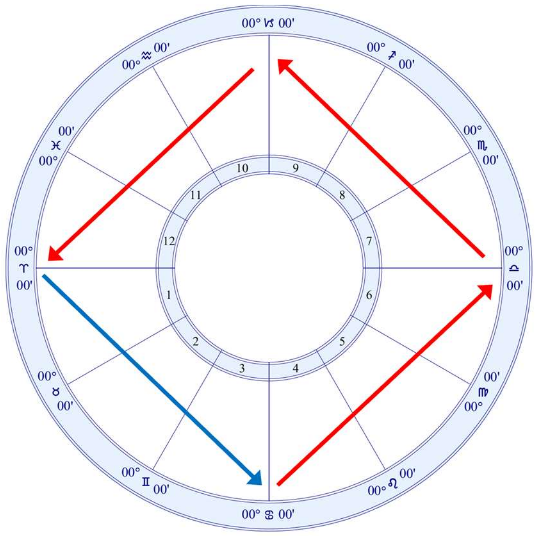
Recall now that, for a counting number *n*, the collection of *n* harmonic angles will frequently be denoted by the ordered set $H_n = \{\theta_n, 2\theta_n, 3\theta_n, \dots, (n-1)\theta_n, 0°\}$. Thus, $H_1 = \{0°\}$, $H_2 = \{180°, 0°\}$, $H_3 = \{120°, 240°, 0°\}$, and $H_4 = \{90°, 180°, 270°, 0°\}$. Furthermore, if $n > 2$, then a regular polygon[3] will be generated if the harmonic angles are plotted around a zodiac wheel and then connected by straight lines in the order that they occur in H_n. Below in Figure 8 are the corresponding graphs for H_3, H_4, and H_{12}, and in each instance the root *n* harmonic angle is shown by a blue line.

Figure 8

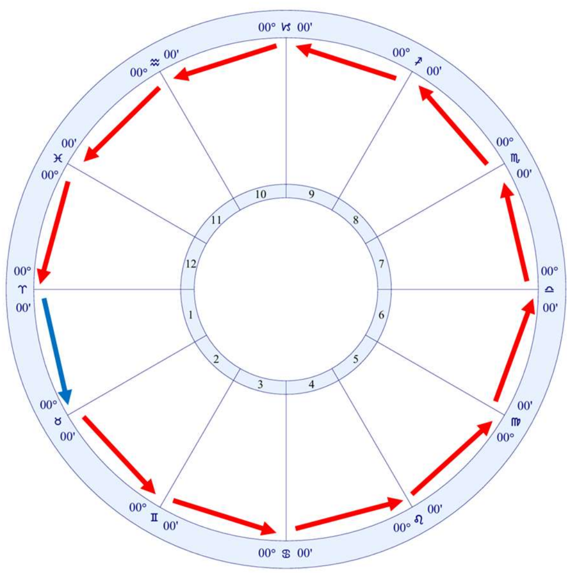
The graphs below give visual representation for H_3, H_4, *and* H_{12}.



$H_3 = \{120°, 240°, 0°\}$



$H_4 = \{90°, 180°, 270°, 0°\}$



$H_{12} = \{30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°, 0°\}$

It should be remarked that because of the above theorem, the 1st House in a *n* harmonic chart always begins with the *n* harmonic of the Ascendant in the natal chart, but then the subsequent house cusps are always $30°$ beyond the previous house cusp. It would, indeed, be nice if one could just find the *n* harmonic of each natal house cusp and use that for house cusps in the *n* harmonic chart, but that does not work. For instance, if one were constructing the 2 harmonic chart, then by Theorem 5, every $180°$ angle would be converted to $0°$ in the 2 harmonic. This means, for example, that the 2nd and 8th natal House cusps would be conjunct in the 2 harmonic chart since they are $180°$ apart in the natal chart. Thus, this approach to determining house cusps clearly does not work, and, instead, one has to use an equal house system.

The Harmonic of a Harmonic for Counting Numbers

The next theorem states that if $k = mn$ where m and n are counting numbers, then the k harmonic is equal to the *m* harmonic of the *n* harmonic. For example, the 6 harmonic chart is identical to the 3 harmonic of the 2 harmonic chart, and similarly, the 6 harmonic chart is identical to the 2 harmonic chart of the 3 harmonic chart. This relationship always holds for counting numbers but not always for other kinds of rational number harmonics. To prove this, a simple lemma needs to first be derived.

Lemma 1: If *n* is a counting number and *q* and *p* are positive rational numbers, then $[n \cdot (q\ modulo\ p)]\ modulo\ p = (nq)\ modulo\ p$.

Proof: Suppose that *r* is the remainder after dividing *q* by *p*. Then $q = w \cdot p + r$ where *w* is a whole number, and it follows that $q\ modulo\ p = r$. From this it follows that $n \cdot (q\ modulo\ p) = nr$. Also, $nq = n(w \cdot p + r) = nw \cdot p + nr$. Hence, $(nq)\ modulo\ p = (\ nw \cdot p + nr)\ modulo\ p$. However, since $nw \cdot p$ is a whole number multiple of p, it follows that $(nq)\ modulo\ p = (\ nw \cdot p + nr)\ modulo\ p = (nr)\ modulo\ p$. Therefore, $[n \cdot (q\ modulo\ p)]\ modulo\ p = (nr)\ modulo\ p = (nq)\ modulo\ p$.

■

Lemma 1 will usually be applied to the case where $p = 360°$. In particular, the conditions of Lemma 1 will be used to conclude that for a counting number *n* and a positive rational number *q* that $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = (nq)\ modulo\ 360°$.

As a concrete example, suppose $n = 100$, $q = \frac{1260}{3.1}$, and $p = 360°$. Then $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = \left[100 \cdot \left(\frac{1260}{3.1}\ modulo\ 360°\right)\right]\ modulo\ 360° = [100 \cdot (46.4516129)]\ modulo\ 360° = (4645.16129)\ modulo\ 360° = 325.16129°$, and $(nq)\ modulo\ 360° = \left(100 \cdot \frac{1260}{3.1}\right) modulo\ 360° = 40645.16129\ modulo\ 360° = 325.16129°$.

Theorem 6: Given any angle θ such that $0° \leq \theta < 360°$, and any counting number $k = mn$ where m and n are also counting numbers, then the k harmonic chart is identical to the m harmonic chart of the n harmonic chart.

Proof: Recall that to convert a natal angle θ, with $0° \leq \theta < 360°$, to its corresponding angle in the k, m, or n harmonic charts for counting numbers k, m, or n, respectively, one need only to multiply the angle θ by $k, m,$ or n and express the result $modulo\ 360°$. But by Lemma 1, $(k \cdot \theta)\ modulo\ 360°. = (mn \cdot \theta)\ modulo\ 360° = [m \cdot (n \cdot \theta\ modulo\ 360°)]\ modulo\ 360°$. However, this last expression is the *m* harmonic of the *n* harmonic, and so it has been shown that if $k = mn$ where m and n are counting numbers, then the *k* harmonic is equal to the *m* harmonic of the *n* harmonic.

■

Corollary 6.1: Given any angle θ such that $0° \leq \theta < 360°$, and any counting number $k = mn$ where m and n are also counting numbers, then the m harmonic chart of the n harmonic is equal to the *n* harmonic of the *m* harmonic.

Proof: This follows immediately from Theorem 6 since both are equal to the *k* harmonic.

■

As previously stated, Theorem 6 and Corollary 6.1 do not hold true for all positive rational number harmonics as the examples below show:

1. $(2.3 \cdot 1.3 \cdot 330°)\ modulo\ 360° = \mathbf{266.7}°$,
2. $[2.3(1.3 \cdot 330°\ modulo\ 360°)]\ modulo\ 360° = \mathbf{158.7}°$, and
3. $[1.3(2.3 \cdot 330°\ modulo\ 360°)]\ modulo\ 360° = \mathbf{50.7}°$.

Similarly, Theorem 6 does not hold for all irrational numbers since:

1. $(\pi \cdot \sqrt{2} \cdot 330°)\ modulo\ 360° \approx \mathbf{26.15}°$,
2. $[\pi \cdot (\sqrt{2} \cdot 330°\ modulo\ 360°)]\ modulo\ 360° \approx \mathbf{335.18}°$, and
3. $[\sqrt{2} \cdot (\pi \cdot 330°\ modulo\ 360°)]\ modulo\ 360° \approx \mathbf{87.92}°$.

On the other hand, for the counting number product 6 = (2)(3) and as guaranteed by the theorem above, one has:

1. $(6 \cdot 300°)\ modulo\ 360° = 1800°\ modulo\ 360° = \mathbf{0}°$,
2. $[2 \cdot (3 \cdot 300°\ modulo\ 360°)]\ modulo\ 360° = [2 \cdot (900°\ modulo\ 360°)]\ modulo\ 360° = (2 \cdot 180°)\ modulo\ 360° = 360°\ modulo\ 360° = \mathbf{0}°$, and
3. $[3 \cdot (2 \cdot 300°\ modulo\ 360°)]\ modulo\ 360° = [3 \cdot (600°\ modulo\ 360°)]\ modulo\ 360° = (3 \cdot 240°)\ modulo\ 360° = 720°\ modulo\ 360° = \mathbf{0}°$.

Positive Rational Harmonics and Their Relationship to Counting Number Harmonics

Even though Theorem 6 does not hold for all positive rational numbers, there are, nonetheless, important connections between counting number harmonics and other positive rational number harmonics as the theorems below will show. To explain the main result in Theorem 7, suppose one starts with a rational number such as 1.5. Then like all rational numbers, this number can also be written as a fraction, specifically $\frac{3}{2}$. Hence, the root 1.5 or $\frac{3}{2}$ harmonic angle is $\frac{360°}{1.5} = \frac{360°}{3/2} = 240°$. However, recall that the way to simplify a complex fraction like $\frac{360°}{3/2}$ is to flip the denominator and multiply. This results in $\frac{360°}{3/2} = 2 \cdot \frac{360°}{3} = 2 \cdot 120° = 240°$. In other words, the root $\frac{3}{2}$ harmonic angle, $\frac{360°}{3/2}$, is equal to 2 times the root 3 harmonic angle, and this suggests that for any positive rational number $\frac{n}{m}$, the root $\frac{n}{m}$ angle is just *m* times the root *n* harmonic angle. And that is exactly what is proven in Theorem 7.

Theorem 7: If $\frac{n}{m}$ is a positive rational number in lowest terms where *n* and *m* are counting numbers, then the root $\frac{n}{m}$ harmonic angle is also a *n* harmonic angle equal to $\left(m \cdot \frac{360°}{n}\right)$ $modulo$ $360°$.

Proof: If $n > 1$, then note that $\frac{360°}{n} < 360°$ is the root *n* harmonic angle, and since the root $\frac{n}{m}$ harmonic angle is $\left(\frac{360°}{n/m}\right) modulus\ 360° = \left(m \cdot \frac{360°}{n}\right)$ $modulo\ 360°$, the result that the root $\frac{n}{m}$ harmonic angle is also a *n* harmonic angle immediately follows. On the other hand, if $n = 1$, then the root *n* harmonic angle is $\frac{360°}{1}$ $modulo\ 360° = 360°\ modulo\ 360° = 0°$ and the root $\frac{1}{m}$ harmonic angle is $\left(\frac{360°}{1/m}\right) modulus\ 360° = (m \cdot 360°)\ modulo\ 360° = 0°$ is also a *n* harmonic angle equal to $\left(m \cdot \frac{360°}{1}\right)$ $modulo\ 360°$..

■

It should also be noted that since every positive rational number can be written in the form $\frac{n}{m}$ where *n* and *m* are counting numbers and $\frac{n}{m}$ is in lowest terms, the above theorem applies to all positive rational numbers. Hence, under these conditions, the $\frac{n}{m}$ root harmonic angle is also a *n* harmonic angle. The next theorem (Theorem 8) will take things further by showing that the $\frac{n}{m}$ harmonic angles in their natural order (i.e. $H_{n/m}$) are just a permutation of the natural ordering for the set of *n* harmonic angles (i.e. H_n). In particular, one of the diagrams (Figure 9) following the theorems will show geometrically that the ordered set of $\frac{12}{5}$ harmonic angles,
$H_{12/5} = \{150°, 300°, 90°, 240°, 30°, 180°, 330°, 120°, 270°, 60°, 210°, 0°\}$, is just a permutation of the usual ordering for the set of 12 harmonic angles, $H_{12} = \{30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°, 0°\}$.

Theorem 8: If $\frac{n}{m}$ is a positive rational number in lowest terms where *n* and *m* are counting numbers, then multiplying the root $\frac{n}{m}$ harmonic angle in turn by the numbers $1, 2, 3, \ldots, n$ creates a permutation of the usual ordering for the *n* harmonic angles. In other words, a permutation of $\theta_n = \frac{360°}{n}, 2 \cdot \frac{360°}{n}, 3 \cdot \frac{360°}{n}, \ldots, \left(n \cdot \frac{360°}{n}\right)$ $modulo\ 360° = 0°$.

Proof: Let $\theta_{n/m} = \left(\frac{360°}{n/m}\right) modulo\ 360° = \left(m \cdot \frac{360°}{n}\right)$ $modulo\ 360°$ where $\frac{n}{m}$ is in lowest terms. Then note that as shown in Theorem 7, θ_n is both a $\frac{n}{m}$ root harmonic angle and also a *n* harmonic angle. Consequently, every counting number multiple of $\theta_{n/m}$ reduced $modulo\ 360°$ is also both a $\frac{n}{m}$ harmonic angle and a *n* harmonic angle. Since $\theta_{n/m}$ is also a *n* harmonic angle, it follows from Theorem 5 that $\left(n \cdot \theta_{n/m}\right) modulo\ 360° = 0°$. Now suppose that *t* is the smallest counting number such that $\left(t \cdot \theta_{n/m}\right) modulo\ 360° = 0°$. Then clearly $t \leq n$. If $t < n$, then $0° = \left(t \cdot \theta_{n/m}\right) modulo\ 360° = \left(t \cdot m \cdot \frac{360°}{n}\right)$ $modulo\ 360°$. But this implies that *n* divides into $t \cdot m$, and this in turn implies that a nontrivial factor of *t* times a nontrivial factor of *m* is equal to *n*. However, this last statement implies that

m and n contain a nontrivial common factor which, in turn, contradicts our assumption that $\frac{n}{m}$ is in lowest terms. Hence, $t = n$. Thus, multiplying the root $\frac{n}{m}$ harmonic angle $\theta_{n/m}$ by the numbers 1, …, n results in n distinct $\frac{n}{m}$ harmonic angles which one can list as $\theta_{n/m}, (2\theta_{n/m})\ modulo\ 360°, (3\theta_{n/m})\ modulo\ 360°, \ldots (n\theta_{n/m})\ modulo\ 360° = 0°$. But again, since each $\frac{n}{m}$ harmonic angle is also a n harmonic angle, it follows that the n distinct $\frac{n}{m}$ harmonic angles must be the same as the n distinct n harmonic angles but in a possibly different order when $(k \cdot \theta_{n/m})\ modulo\ 360°$ is computed in the order corresponding to $k = 1, \ldots, n$. Therefore, the $\frac{n}{m}$ harmonic angles in their natural order are simply a permutation of the n harmonic angles

■

Corollary 8.1: If m and n are counting numbers and if $\frac{n}{m}$ is in lowest terms, then the number of $\frac{n}{m}$ harmonic angles is equal to n (the number of n harmonic angles), and every $\frac{n}{m}$ harmonic angle is also a n harmonic angle.

Proof: This is just a formal re-statement of some of the things that came to light in proving Theorem 8.

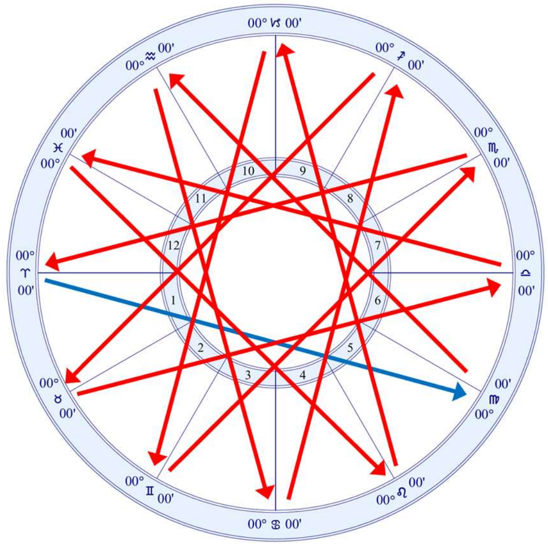
■

Recall that the set of n harmonic angles in their natural order beginning with the root n harmonic angle and ending with 0° is denoted by H_n while the set of $\frac{n}{m}$ harmonic angles in their natural order is denoted by $H_{n/m}$. As ordered sets, these two contain the same elements but in possibly different orders. One consequence of this is that counting number harmonics may be all that matter since every $\frac{n}{m}$ harmonic angle is also a n harmonic angle.

Recall, too, that, when graphed on a $360°$ circle, the n harmonic angles define a regular n-sided polygon for $n > 2$. In a similar manner, if $\frac{n}{m}$ is in lowest terms and $m \neq 1$, then the corresponding graphs of the $\frac{n}{m}$ harmonic angles in $H_{n/m}$ will often and with few exceptions will form a star shape called a star polygon. Examples are shown below in Figure 9 for $H_{12/5}$ and $H_{12/7}$, and the root harmonic angle is indicated by a blue arrow. Also, for each case, $H_{12/}$ and $H_{12/7}$, the natural ordering of the angles is just a permutation of the natural ordering for the angles of H_{12}.

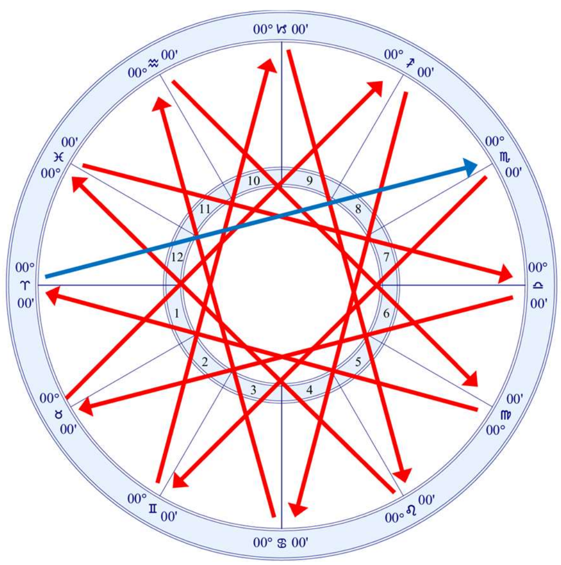
Figure 9

The graphs below show visually how $H_{12/5}$ and $H_{12/7}$ create permutations of the elements of H_{12}.



$H_{12/5} = \{150°, 300°, 90°, 240°, 30°, 180°, 330°, 120°, 270°, 60°, 210°, 0°\}$

$H_{12} = \{30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°, 0°\}$



$H_{12/7} = \{210°, 60°, 270°, 120°, 330°, 180°, 30°, 240°, 90°, 300°, 150°, 0°\}$

$H_{12} = \{30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°, 0°\}$

One more theorem to be proven about positive rational number harmonics will assert the equality between, for example, the root $\frac{3}{5}$ harmonic angle and the root $\frac{3}{2}$ harmonic angle. Of course, for particular cases like this one can always show equality through direct calculation. For instance, $\left(\frac{360°}{3/5}\right)$ $modulo$ $360° = \left(5 \cdot \frac{360°}{3}\right)$ $modulo$ $360° = (5 \cdot 120°)$ $modulo$ $360° = 600$ $modulo$ $360° = 240°$ and $\left(\frac{360°}{3/2}\right)$ $modulo$ $360° = \left(2 \cdot \frac{360°}{3}\right)$ $modulo$ $360° = (2 \cdot 120°)$ $modulo$ $360° = 240°$ $modulo$ $360° = 240°$. Furthermore, since the root harmonic angles associated with $\frac{3}{5}$ and $\frac{3}{2}$ are the same, the same natural ordering will occur as one multiplies each root harmonic angle by successive counting numbers. In other words, $H_{3/5} = H_{3/2}$.

Theorem 9: If $n > 1$ and *m* are counting numbers with $\frac{n}{m}$ in lowest terms and if $m = pn + r$ where *p* and *r* are whole numbers and $r < n$, then $r \neq 0$ and $\left(\frac{360°}{n/m}\right) modulo\ 360° = r \cdot \frac{360°}{n} = \frac{360°}{n/r} = \left(\frac{360°}{n/r}\right) modulo\ 360° = \left(\frac{360°}{n/(m\ modulo\ n)}\right) modulo\ 360°$.

Proof: Clearly, $\left(\frac{360°}{n/m}\right) modulo\ 360° = \left(m \cdot \frac{360°}{n}\right) modulo\ 360° = \left([pn + r] \cdot \frac{360°}{n}\right) modulo\ 360° = \left(p \cdot 360° + r \cdot \frac{360°}{n}\right) modulo\ 360° = \left(r \cdot \frac{360°}{n}\right) modulo\ 360° = r \cdot \frac{360°}{n}$ since $0° \leq r \cdot \frac{360°}{n} < 360°$. Furthermore, since $\frac{n}{m}$ is in lowest terms with $n > 1$, it follows that *m* is not divisible by *n* and, hence, $r \neq 0$. Consequently, $\left(\frac{360°}{n/m}\right) modulo\ 360° = r \cdot \frac{360°}{n} = \frac{360°}{n/r} = \left(\frac{360°}{n/r}\right) modulo\ 360° = \left(\frac{360°}{n/(m\ modulo\ n)}\right) modulo\ 360°$.

■

The theorem above shows that if $\frac{n}{m}$ is a rational number in lowest terms with $m > n$ and if *r* is the nonzero remainder when *m* is divided by *n*, then the root $\frac{n}{m}$ harmonic angle is the same as the root $\frac{n}{r}$ harmonic angle. Another way to denote this is by saying that $H_{n/m} = H_{n/r}$. In particular, $H_{3/5} = H_{3/2}$ since 5 divided by 3 yields a remainder of 2, and $H_{3/4} = H_{3/1} = H_3$ since 4 divided by 3 yields a remainder of 1.

One very important consequence of Theorem 9 is that in the study of positive rational number harmonics, one can always replace a rational number between 0 and 1, such as $\frac{3}{5}$, by a corresponding rational number greater than 1, such as $\frac{3}{2}$, that yields the same set of harmonic angles. The advantage of this is that if *r* is a positive rational number that is greater than 1, then $\frac{360°}{r}$ is automatically less than 360° and, thus, there is no need to further reduce the result $modulo$ 360°. This fact will be used in some of the proofs that follow.

Yet another consequence of Theorem 9 is that it gives us another way to look at Lemma 1. Recall that this lemma says:

If n is a counting number and q and p are positive rational numbers, then $[n \cdot (q\ modulo\ p)]\ modulo\ p = (nq)\ modulo\ p$.

Recall, too, that the only value this paper is interested in for *p* is 360°, and suppose, too, that $q = \frac{360°}{r}$ where *r* is a positive rational number. Then one can now further analyze this lemma by considering the following three cases.

Case 1 (*r* = 1): In this case, $q = \frac{360°}{1} = 360°$ and $q\ modulo\ 360° = \frac{360°}{r}\ modulo\ 360° = \frac{360°}{1}\ modulo\ 360° = 0°$, and it is now trivial that $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = (n \cdot 0°)\ modulo\ 360° = 0° = (n \cdot 360°) modulo\ 360°° = (n \cdot q) modulo\ 360°$.

Case 2 ($r > 1$): In this case, $q = \frac{360°}{r}$ and $q\ modulo\ 360° = \frac{360°}{r}\ modulo\ 360° = \frac{360°}{r} = q$ since $q = \frac{360°}{r} < 360°$. Thus, $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = \left[n \cdot \left(\frac{360°}{r}\ modulo\ 360°\right)\right]\ modulo\ 360° = \left(n \cdot \frac{360°}{r}\right)\ modulo\ 360° = (n \cdot q)\ modulo\ 360°$.

Case 3 ($0 < r < 1$): In this case, $q = \frac{360°}{r} > 360°$. However, using the method outlined in Theorem 9, one can find $r' > 1$ such that $q' = \frac{360°}{r'} < 360°$ and $q\ modulo\ 360° = \frac{360°}{r}\ modulo\ 360° = \frac{360°}{r'}\ modulo\ 360° = q' modulo\ 360°$. Things now revert back to case 2 by noting that $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = [n \cdot (q'\ modulo\ 360°)]\ modulo\ 360° = (n \cdot q') modulo\ 360°$. However, things can be taken one step further. Since $q' = \frac{360°}{r'} < 360°$, it is clear that is clear that $q' modulo\ 360° = q'$. Also, since $q' = q' modulo\ 360° = q\ modulo\ 360°$ and since $q = \frac{360°}{r} > 360°$, it must follow that $q = w \cdot 360° + q'$. Hence, $(nq)\ modulo\ 360° = [n(w \cdot 360° + q')]\ modulo\ 360° = (nw \cdot 360° + nq') modulo\ 360° = (nq')\ modulo\ 360°$. Thus, in this case, too, it is true that $[n \cdot (q\ modulo\ 360°)]\ modulo\ 360° = (nq)\ modulo\ 360°$.

Cancellation Theorems

A few of what might be called cancellation theorems will now be proven. The first one guarantees that since, for example, $6 = 2 \cdot 3$, it will follow that 2 times a 6 harmonic angle, reduced *modulo* 360°, is a 3 harmonic angle, and 3 times a 6 harmonic angle, reduced *modulo* 360°, is a 2 harmonic angle. For example, taking the 6 harmonic angle of 60°, then $2 \cdot 60° = 120°$ (a 3 harmonic angle), and $3 \cdot 60° = 180°$ (a 2 harmonic angle).

Theorem 10: If θ is a *r* harmonic angle and if $r = nq$ where *n* is a counting number and *q* is a positive rational number, then $(n \cdot \theta)\ modulo\ 360°$is a *q* harmonic angle.

Proof: Since θ is a *r* harmonic angle, it follows that $\theta = (k \cdot \theta_r)\ modulo\ 360° = \left(k \cdot \frac{360°}{r}\right)\ modulo\ 360°$ for some counting number *k* and where θ_r is the root *r* harmonic angle. Hence, if $r = nq$, then by Lemma 1, $(n \cdot \theta)\ modulo\ 360° = [n \cdot ([k \cdot \theta_r]\ modulo\ 360°)]\ modulo\ 360° = (n \cdot k \cdot \theta_r)\ modulo\ 360° = \left(n \cdot k \cdot \frac{360°}{r}\right)\ modulo\ 360° = \left(n \cdot k \cdot \frac{360°}{nq}\right)\ modulo\ 360° = \left(k \cdot \frac{360°}{q}\right)\ modulo\ 360°$. Therefore, $(n \cdot \theta)\ modulo\ 360°$ is a *q* harmonic angle.

■

Corollary 10.1: If $\theta_r = \left(\frac{360°}{r}\right)\ modulo\ 360°$ is the root *r* harmonic angle and if $r = nq$ where *n* is a counting number and *q* is a positive rational number, then $(n \cdot \theta_r)\ modulo\ 360°$ is the root *q* harmonic angle.

Proof: One can simply repeat the proof for Theorem 10 with $k = 1$.

■

And now, here is another cancellation theorem that this time involves both addition and multiplication. This theorem shows things such as if one multiplies the sum of 2 and 3 harmonic angles by 2, then the result reduced $modulo$ 360° is a 3 harmonic angle, and if one multiplies that same sum by 3, then one is left after reduction $modulo$ 360° with only a 2 harmonic angle. Essentially, multiplication of a 2 harmonic angle by 2 cancels out the 2 harmonic angle, and multiplication of a 3 harmonic angle by 3 cancels out the 3 harmonic angle. For example, $[2 \cdot (180° + 120°)]\ modulo\ 360° =$, $[360° + 240°]\ modulo\ 360° = 240°$, and $[3 \cdot (180° + 120°)]\ modulo\ 360° =$, $[540° + 360°]\ modulo\ 360° = 540°\ modulo\ 360° = 180°$

Theorem 11: If an angle θ can be written as $\theta = m\theta_r + n\theta_q$ where θ_r and θ_q are the root r and q harmonic angles, respectively, for positive rational numbers r and q that are both greater than 1 and if m and n are counting numbers, then $(r\theta)\ modulo\ 360° = (r \cdot n\theta_q)\ modulo\ 360°$.

Proof: Since $(r \cdot m\theta_r)\ modulo\ 360° = \left(r \cdot m \cdot \frac{360°}{r}\right)\ modulo\ 360° = (m \cdot 360°)\ modulo\ 360° = 0°$, it follows that $(r\theta)\ modulo\ 360° = (r \cdot m\theta_r + r \cdot n\theta_q)\ modulo\ 360° = (r \cdot n\theta_q)\ modulo\ 360°$.

■

Consequences of these cancellation theorems are that 2 times any 6 harmonic angle will be a 3 harmonic angle, after reduction $modulo$ 360°, since $2 \cdot \frac{360°}{6} = \frac{360°}{3}$, and 4 times the sum of a 4 harmonic angle and a 3 harmonic angle computed $modulo$ 360° will be equal to 4 times the 3 harmonic angle computed $modulo$ 360°. An example of the latter is $[4 \cdot 210°]\ modulo\ 360° = [4 \cdot (90° + 120°)]\ modulo\ 360° = [4 \cdot 120°]\ modulo\ 360° = 120°$.

Sums of Angles

And now before going any further, a general theorem regarding the sum of two angles will be proven. This theorem says that a counting number multiple of the sum or difference of r and q harmonic angles, for positive rational numbers r and q, will still be a sum or difference of r and q harmonic angles prior to the final reduction of that sum or difference $modulo$ 360°

Theorem 11: If an angle θ can be written as $\theta = k_r\theta_r \pm k_q\theta_q$ where θ_r, θ_q are r and q root harmonic angles for positive rational numbers r and q and counting numbers k_r and k_q, respectively, and if n is also a counting number, then $(n\theta)\ modulo\ 360° = \left[(n \cdot k_r\theta_r)\ modulo\ 360° \pm (n \cdot k_q\theta_q)\ modulo\ 360°\right]\ modulo\ 360°$.

Proof: Clearly, $n\theta = n(k_r\theta_r \pm k_q\theta_q) = n \cdot k_r\theta_r \pm n \cdot k_q\theta_q = (w_r \cdot 360° + \theta_{R_r}) \pm (w_q \cdot 360° + \theta_{R_q})$ where w_r and w_q are whole numbers and θ_{R_r} and θ_{R_q} are the remainders after dividing $n \cdot k_r\theta_r$ and $n \cdot k_q\theta_q$, respectively, by 360°. Notice, too, that $(n \cdot k_r\theta_r)\ modulo\ 360° = (w_r \cdot 360° +$

$\theta_{R_r}) \; modulo \; 360° = \theta_{R_r}$ and $\left(n \cdot k_q \theta_q\right) \; modulo \; 360° = \left(w_q \cdot 360° + \theta_{R_q}\right) \; modulo \; 360° = \theta_{R_q}$
Therefore, $(n\theta) \; modulo \; 360° = \left(n \cdot k_r \theta_r \pm n \cdot k_q \theta_q\right) \; modulo \; 360° = \left[\left(w_r \cdot 360° + \theta_{R_r}\right) \pm \left(w_q \cdot 360° + \theta_{R_q}\right)\right] \; modulo \; 360° = \left(\theta_{R_r} \pm \theta_{R_q}\right) \; modulo \; 360° = \left[(n \cdot k_r \theta_r) \; modulo \; 360° \pm \left(n \cdot k_q \theta_q\right) \; modulo \; 360°\right] \; modulo \; 360°$.

■

For an application of Theorem 11, suppose that two planets are $125°$ apart. Then most astrologers would probably say that the two planets are trine one another with a $5°$ orb. Thus, one can write $125° = 120° + 5°$ where $5°$ is also the root 72 harmonic angle, and any counting number harmonic of the angle $125°$ can be thought of, before reduction $modulo \; 360°$, as the sum of a multiple of a 3 harmonic angle and a multiple of a 72 harmonic angle. In this example, the error term (the orb) will always be based on a 72 harmonic angle just so long as a counting number harmonic of $125°$ is being computed.

Fixed Points

When it was noted that all higher harmonics of 0° remain at 0°, a quantity that remains fixed in higher harmonics was essentially being identified. Such fixed points are often of interest in mathematics and below is another theorem that identifies points that will remain fixed in certain higher harmonics. Theorem 12 essentially states that any *n* harmonic angle will stay fixed in the *n* + 1 harmonic. Thus, for example, a 6 harmonic angle such as $60°$ will remain unchanged in the 7 harmonic, and one can show this by the direct calculation $(7 \cdot 60°) \; modulo \; 360° = 420° \; modulo \; 360° = 60°$.

Theorem 12: If θ is an angle with $0° < \theta < 360°$ and if *n* is a counting number, then θ is a n harmonic angle if and only if $[(n+1)\theta] \; modulo \; 360° = \theta$. In other words, θ is a n harmonic angle for some counting number *n* if and only if θ is a fixed point in the transition to a $(n+1)$ harmonic chart.

Proof: (⇒) If θ is a n harmonic angle, then $\theta = \left[k\left(\frac{360°}{n}\right)\right] \; modulo \; 360°$ where k is a counting number.
Thus, using Lemma 1 again, $[(n+1)\theta] \; modulo \; 360° = \left[(n+1) \cdot \left(\left[k\left(\frac{360°}{n}\right)\right] \; modulo \; 360°\right)\right] \; modulo \; 360° = \left[(n+1) \cdot k\left(\frac{360°}{n}\right)\right] modulo \; 360° = \left[k(360°) + k\left(\frac{360°}{n}\right)\right] modulo \; 360° = \left[k\left(\frac{360°}{n}\right)\right] \; modulo \; 360° = \theta.$

(⇐) If θ with $0° < \theta < 360°$ is a fixed point in the transition to the $(n+1)$ harmonic, then $[(n+1)\theta] \; modulo \; 360° = \theta$ which implies that $[n\theta + \theta] \; modulo \; 360° = \theta$. Since $0° < \theta < 360°$, the remainder when θ is divided by 360 is θ which implies that that the remainder when $n\theta$ is divided by 360 is 0° since otherwise if $n\theta = w \cdot 360° + \theta'$ where *w* is a whole number and $0° < \theta' < 360°$, then $[(n+1)\theta] \; modulo \; 360° = (n\theta + \theta) \; modulo \; 360° = (w \cdot 360° + \theta' + \theta) \; modulo \; 360° = (\theta' + \theta) \; modulo \; 360° \neq \theta$. To see this more clearly, notice that if $\theta' + \theta < 360°$. then $(\theta' + \theta) \; modulo \; 360° = \theta' + \theta > \theta$, and if $\theta' + \theta > 360°$, then $(\theta' + \theta) \; modulo \; 360° < \theta$ since less than $360°$ has been added to θ. On the other hand, if $\theta' + \theta = 360°$, then $[(n+1)\theta] \; modulo \; 360° = (n\theta + \theta) \; modulo \; 360° = (w \cdot 360° + \theta' + \theta) \; modulo \; 360° = (w \cdot 360° + 360°) \; modulo \; 360° =$

$([w+1]\cdot 360°)\ modulo\ 360° = 0° \neq \theta$ since it was assumed that $0° < \theta < 360°$, and that contradicts our hypothesis that $[(n+1)\theta]\ modulo\ 360° = \theta$. Thus, the assumption above that $0° < \theta' < 360°$ is incorrect, and instead $\theta' = 0°$. Hence, it follows that $n\theta = k(360°)$ for some counting number k. But this implies that $\theta = k\left(\frac{360°}{n}\right) < 360°$, and therefore, θ is a n harmonic angle.

■

While it has been proven in the above theorem that a *n* harmonic angle will remain fixed in the $n+1$ harmonic for a counting number *n*, something even stronger will be shown in the next theorem. Namely that for a counting number *n*, a *n* harmonic angle will remain fixed in any harmonic that is equal to the value of a polynomial with integer coefficients and with constant term 1 when evaluated at *n*. As an example, 72° is a 5 harmonic angle, and if one takes a polynomial with integer coefficients like $p(x) = 4x^3 - 3x^2 + x + 1$ and evaluates this at 5 to get $p(5) = 4\cdot 5^3 - 3\cdot 5^2 + 5 + 1 = 431$, then $(431\cdot 72°)\ modulo\ 360° = 72°$.

Theorem 13: Let $\theta = k\cdot\frac{360°}{n}$ be a n harmonic angle where *n* and *k* are counting numbers and $1 \leq k \leq n-1$. Also, let $p = a_j n^j + a_{j-1}n^{j-1} + \cdots + a_1 n + 1$ be a polynomial with integer coefficients evaluated at *n* and where *j* is a counting number. Then $(p\cdot\theta)\ modulo\ 360° = \theta$.

Proof: Note that $p\cdot\theta = \left(a_j n^j + a_{j-1}n^{j-1} + \cdots + a_1 n + 1\right)\cdot\ k\cdot\frac{360°}{n} = \left(a_j n^j + a_{j-1}n^{-1} + \cdots + a_1 n\right)\cdot k\cdot\frac{360°}{n} + k\cdot\frac{360°}{n} = \left(a_j n^{j-1} + a_{j-1}n^{j-2} + \cdots + a_1\right)\cdot\ k\cdot 360° + k\cdot\frac{360°}{n}$. Hence, $(p\cdot\theta)\ modulo\ 360° = \left[\left(a_j n^j + a_{j-1}n^{j-1} + \cdots + a_1 n + 1\right)\cdot\ k\cdot\frac{360°}{n}\right]\ modulo\ 360° = \left[\left(a_j n^{j-1} + a_{j-1}n^{j-2} + \cdots + a_1\right)\cdot\ k\cdot 360° + k\cdot\frac{360°}{n}\right]\ modulo\ 360° = \left(k\cdot\frac{360°}{n}\right)\ modulo\ 360° = k\cdot\frac{360°}{n} = \theta$.

■

A special case of Theorem 13 is when $\theta = k\cdot\frac{360°}{n}$ is a n harmonic angle, where *n* and *k* are counting numbers with $n > 1$ and $1 \leq k \leq n-1$, and $p = (n+1)^j$ for a counting number *j*. In this case, $p = (n+1)^j$ expands into a polynomial with integer coefficients and constant term 1, and so Theorem 13 applies to give us $\left[(n+1)^j\cdot\theta\right]\ modulo\ 360° = \theta$. As an example, let $n = 4, k = 3$, and $j = 5$. Then $\theta = k\cdot\frac{360°}{n} = 3\cdot\frac{360°}{4} = 3\cdot 90° = 270°$, and $\left[(n+1)^j\cdot\theta\right]\ modulo\ 360° = [(4+1)^5\cdot 270°]\ modulo\ 360° = 843{,}750°\ modulo\ 360° = 270° = \theta$.

The Fundamental Theorem of Harmonic Astrology

The next theorem deserves to be called the Fundamental Theorem of Harmonic Astrology. It establishes a duality between the *least common multiple* of two harmonics and the *greatest common divisor* of the corresponding root harmonic angles. Specifically, it states that for counting number harmonics *n* and *m* and associate root harmonic angles θ_m and θ_n, one has that $gcd(\theta_m, \theta_n) = \frac{360°}{lcm(m,n)}$. As an example, consider the 4 and 6 harmonics with root harmonic angles 90° and 60°, respectively. Then Theorem 14 below insures that $\frac{360°}{lcm(4,6)} = \frac{360°}{12} = 30° = gcd\ (90°, 60°)$.

This theorem will be proven first for counting number harmonics and then extended to include all positive rational number harmonics. First, though, a few more lemmas will have to be shown.

Lemma 2: If *x* and *y* are counting numbers, then the $gcd(x, y) \cdot lcm(x, y) = xy$.

Proof: The $lcm(x, y)$ is the product of all factors that exclusively belong to *x* times all factors that exclusively belong to *y* times all factors that are common to both *x* and *y*. Furthermore, the product of all factors that are common to both *x* and *y* is simply the $gcd(x, y)$. If one now multiplies *x* and *y* together, then this product contains both all factors that exclusively belong to *x* and all factors that exclusively belong to *y*, but it also contains twice all the factors that belong to both *x* and *y*. Hence, if one divides out the $gcd(x, y)$ once, then the result will be equal to the $lcm(x, y)$. In other words, $lcm(x, y) = \frac{xy}{gcd\ (x,y)}$ which implies that $gcd(x, y) \cdot lcm(x, y) = xy$.

■

Lemma 3: If *a, b, c, d,* and *k* are counting numbers, then $gcd\left(\frac{k \cdot a}{b}, \frac{k \cdot c}{d}\right) = \frac{k}{bd} \cdot gcd\ (ad, bc)$.

Proof: By Definition 8, $gcd\left(\frac{k \cdot a}{b}, \frac{k \cdot c}{d}\right) = gcd\left(\frac{kab}{bd}, \frac{kbc}{bd}\right) = \frac{gcd\ (kad, kbc)}{bd} = \frac{1}{bd} \cdot gcd\ (kad, kbc)$. Furthermore, since *k* is a common divisor of both *kad* and *kbc*, it is clear that $gcd(kad, kbc) = k \cdot gcd\ (ad, bc)$. Therefore, $gcd\left(\frac{k \cdot a}{b}, \frac{k \cdot c}{d}\right) = \frac{1}{bd} \cdot gcd(kad, kbc) = \frac{k}{bd} \cdot gcd\ (ad, bc)$.

■

Theorem 14 (The Fundamental Theorem of Harmonic Astrology): Suppose that m and n are counting numbers greater than 1 and that θ_m and θ_n are the corresponding m and n root harmonic angles. Also, denote the *least common multiple* of m and n by $lcm(m, n)$ and the *greatest common divisor* of θ_m and θ_n by $gcd(\theta_m, \theta_n)$. Then $gcd(\theta_m, \theta_n) = \frac{360°}{lcm(m,n)}$.

Proof: Using Lemma 2 and Lemma 3 and definition 8, it follows that $gcd(m, n) \cdot lcm(m, n) = mn \Rightarrow \frac{1}{lcm(m,n)} = \frac{gcd(m,n)}{mn} \Rightarrow \frac{360°}{lcm(m,n)} = \frac{360°}{mn} \cdot gcd(m, n) = 360° \cdot gcd\left(\frac{m}{mn}, \frac{n}{mn}\right) = 360° \cdot gcd\left(\frac{1}{n}, \frac{1}{m}\right) = gcd\left(\frac{360°}{n}, \frac{360°}{m}\right) = gcd(\theta_n, \theta_m)$, and that completes the proof!

■

The corollary to Theorem 14 that is given below extends the result to all positive rational numbers. In particular, for positive rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $gcd\left(\frac{360°}{\frac{a}{b}}, \frac{360°}{\frac{c}{d}}\right) = \frac{360°}{lcm\left(\frac{a}{b}, \frac{c}{d}\right)}$.

Corollary 14.1: Suppose that $a, b, c,$ and d are counting numbers such that $\frac{a}{b}$ and $\frac{c}{d}$ are positive rational numbers greater than 1 and that $\frac{360°}{\frac{a}{b}}$ and $\frac{360°}{\frac{c}{d}}$ are the corresponding root angles. Then $gcd\left(\frac{360°}{\frac{a}{b}}, \frac{360°}{\frac{c}{d}}\right) = \frac{360°}{lcm\left(\frac{a}{b}, \frac{c}{d}\right)}$.

Proof: To prove this, definitions 7 & 8 will be used along with the results from Lemma 2 & Lemma 3, in particular that if x and y are counting numbers, then $gcd(x, y) \cdot lcm(x, y) = xy$ which implies that $gcd(x, y) = \frac{xy}{lcm(x,y)}$. Thus, on the one hand, $\frac{360°}{lcm\left(\frac{a}{b}, \frac{c}{d}\right)} = \frac{360°}{lcm\left(\frac{ad}{bd}, \frac{bc}{bd}\right)} = \frac{360°}{\frac{lcm(ad,bc)}{bd}} = \frac{360° \cdot bd}{lcm(ad,bc)}$. But on the other hand, $gcd\left(\frac{360°}{\frac{a}{b}}, \frac{360°}{\frac{c}{d}}\right) = 360° \cdot gcd\left(\frac{b}{a}, \frac{d}{c}\right) = 360° \cdot gcd\left(\frac{bc}{ac}, \frac{ad}{ac}\right) = 360° \cdot \frac{gcd\,(bc,ad)}{ac} = 360° \cdot \frac{\frac{bcad}{lcm(bc,ad)}}{ac} = \frac{360° \cdot bd}{lcm(bc,ad)} = \frac{360°}{\frac{lcm(bc,ad)}{bd}} = \frac{360°}{lcm\left(\frac{bc}{bd}, \frac{ad}{bd}\right)} = \frac{360°}{lcm\left(\frac{c}{d}, \frac{a}{b}\right)} = \frac{360°}{lcm\left(\frac{a}{b}, \frac{c}{d}\right)}$, and this proves the equality.

■

As an example of the above extension of Theorem 14 to include positive rational numbers, consider the positive rational number harmonics $\frac{4}{3}$ and $\frac{6}{5}$ with root harmonic angles $270°$ and $300°$, respectively. Then $\frac{360°}{lcm\left(\frac{4}{3}, \frac{6}{5}\right)} = \frac{360°}{lcm\left(\frac{20}{15}, \frac{18}{15}\right)} = \frac{360°}{\frac{lcm(20,18)}{15}} = \frac{360°}{\frac{180}{15}} = 30° = gcd(270°, 300°) = gcd\left(\frac{360°}{\frac{4}{3}}, \frac{360°}{\frac{6}{5}}\right)$.

The Small Product and the Large Product of Two Harmonics

Next are definitions for two products that can be used simultaneously with harmonics, harmonic angles, and the ordered harmonic sets H_r, and these definitions will add more algebraic structure to the field of harmonics that can be exploited by future mathematical explorations. The first product is called the *small product*, and it is denoted by the symbol "⊠." Specifically, for two positive rational number harmonics, *m* and *n*, with corresponding root harmonic angles $\theta_m = \frac{360°}{m}$ and $\theta_n = \frac{360°}{n}$, the following definition is made.

Definition 13: For positive rational numbers *n* and *m*, root harmonic angles $\theta_m = \frac{360°}{m}$ and $\theta_n = \frac{360°}{n}$, and ordered harmonic sets H_m and H_n, the following *small products* are defined.

1. $m \boxtimes n = lcm(m, n)$
2. $\theta_m \boxtimes \theta_n = gcd(\theta_m, \theta_n) = \frac{360°}{lcm(m,n)}$
3. $H_m \boxtimes H_n = H_{lcm(m,n)}$

Thus, for example,

1. $4 \boxtimes 6 = lcm(4,6) = 12$
2. $\theta_4 \boxtimes \theta_6 = 90° \boxtimes 60° = gcd(90°, 60°) = 30°$
3. $H_4 \boxtimes H_6 = H_{lcm(4,6)} = H_{12}$

A shortcoming of this definition, however, is that $H_2 \boxtimes H_2 = H_{lcm(2,2)} = H_2$ instead of the perhaps expected H_4. Consequently, a second product, called the *large product* of two ordered harmonic sets, is defined by $H_m \otimes H_n = H_{mn}$.

Definition 14: : For positive rational numbers *n* and *m*, root harmonic angles $\theta_m = \frac{360°}{m}$ and $\theta_n = \frac{360°}{n}$, and ordered harmonic sets H_m and H_n, the following *large products* are defined.

1. $m \otimes n = mn$
2. $\theta_m \otimes \theta_n = \theta_{mn} = \frac{360°}{mn}$
3. $H_m \otimes H_n = H_{mn}$

And as an example,

1. $4 \otimes 6 = 24$
2. $\theta_4 \otimes \theta_6 = \theta_{24} = \frac{360°}{24} = 15°$
3. $H_4 \otimes H_6 = H_{24}$

Below are a few results involving the ordered harmonic sets H_r of *r* harmonic angles. In particular, if every *q* harmonic angle is also a *r* harmonic angle, then the notation $H_q \leq H_r$ will be used to indicate this. For example, with $H_3 = \{120°, 240°, 0°\}$ and $H_6 = \{60°, 120°, 180°, 240°, 300°, 0°\}$, it follows that $H_3 \leq H_6$.

The following theorem is given without proof. The proof follows immediately from results about cyclic groups that can be found in group theory (a branch of abstract algebra), but to keep things simple, straying too far into other branches of mathematics will be avoided in this paper.

Theorem 15: If *m* and *n* are counting numbers, then $H_m \leq H_n$ if and only if *m* is a divisor of *n*.

Note that Theorem 15 excludes 0 since 0 is not a counting number, but, nonetheless, $H_0 \leq H_n$ for any counting number *n* since $H_0 = \{0°\}$ is always a subset of H_n, i.e. the element 0° of H_0 is contained within H_n for any counting number *n*. Also, as a consequence of Theorem 15, it can be stated that $H_2 \leq H_6$ since 2 divides evenly into 6. Furthermore, one can define $\frac{H_n}{H_m} = H_{n/m}$. Thus, $\frac{H_6}{H_2} = H_{6/2} = H_3$ and $H_2 \otimes H_3 = H_6$.

Again, when one is first introduced to harmonic charts, it seems like the new locations for astrological chart points in any given harmonic chart are entirely random. However, the theorems above show that just the opposite is true. Instead of being chaotic and random, harmonic charts are highly structured. For example, any counting number times a *r* harmonic angle $modulo$ 360° will result in just another *r* harmonic angle.

Thus, if a 3 harmonic angle such as $120°$, is multiplied by 100, then the result $modulo$ $360°$ has to be either $120°, 240°$ or $0°$. In this case, $(100 \cdot 120°)$ $modulo$ $360° = 120°$, $(101 \cdot 120°)$ $modulo$ $360° = 240°$, and $(102 \cdot 120°)$ $modulo$ $360° = 0°$. If someone now multiplies $102 \cdot 120°$ by any other counting number and simplifies the result $modulo$ $360°$, then they will always get back $0°$ as in the case $(5 \cdot 102 \cdot 120°)$ $modulo$ $360° = (61{,}200°)$ $modulo$ $360° = 0°$. Also, note that if one multiplies any 12 harmonic angle by 3, then the result will be a 4 harmonic angle since $\frac{12}{3} = 4$. For example, $(3 \cdot 210°)$ $modulo$ $360° = (630°)$ $modulo$ $360° = 270°$ which is a 4 harmonic angle, and from this point on any counting number multiple of $270°$ will yield another 4 harmonic angle when simplified $modulo$ $360°$.

Furthermore, if $H_3 = \{120°, 240°, 0°\}$, then $2H_3$ is defined as $\{2 \cdot 120°, 2 \cdot 240°, 2 \cdot 0°\}$ $modulo$ $360° = \{240°, 120°, 0°\}$. Notice, too, that $2H_3 = H_{3/2}$. Similarly, one can verify that $4 \cdot H_{12} = H_{12/4} = H_3$. Additionally, notice that $4H_3 = 4 \cdot \{120°, 240°, 0°\} = \{4 \cdot 120°, 4 \cdot 240°, 4 \cdot 0°\}$ $modulo$ $360° = \{480°, 960°, 0°\}$ $modulo$ $360° = \{120°, 240°, 0°\} = H_3$ which is exactly what is expected since Theorem 12 proved that the 4 harmonic of any 3 harmonic angle will return exactly the same 3 harmonic angles.

Now, the following formal definition is made for expressions of the form rH_q.

Definition 15; If r and q are positive rational numbers and if $H_q = \{\theta_q, 2\theta_q, 3\theta_q, \ldots, 0°\}$, then $rH_q = \{r \cdot \theta_q, r \cdot 2\theta_q, r \cdot 3\theta_q, \ldots, r \cdot 0°\}$ $modulo$ $360°$.

Note that if this multiplication produces duplicate angles, then such results should be collapsed down to a single occurrence of that angle.

Discussion:

From the theorems above, one can see that harmonics are highly structured, and by understanding the mathematics behind harmonics, it is much easier to understand and predict what is going on. Furthermore, the above theorems provide a mathematical model for the universe in which counting number and positive rational number harmonic angles live, and one can now derive additional facts about the mathematics of harmonics by proving more theorems using this mathematical model. Also, in practice most of the aspects one encounters will not be exact. For example, suppose two planets are trine with an orb *modulo* 360 of 4°. Specifically, suppose that the two planets are $124°$ apart. Then one can think of $124° = 120° + 4°$ as the sum of a 3 harmonic angle and a 90 harmonic angle, and any counting number multiple of $124°$ reduced $modulo$ $360°$ will also be the sum of a 3 harmonic angle and a 90 harmonic angle with that sum reduced $modulo$ $360°$. Below are non-technical summaries of many of the results above.

- Even though harmonics can be defined for negative numbers and irrational numbers, those topics have been avoided here since that would overcomplicate this paper. That decision is justified on the grounds that every negative harmonic can be expressed as a corresponding positive harmonic and every irrational number can be approximated as closely as one would like by a rational number. Nonetheless, for the sake of completeness, the mathematical study of harmonics should eventually be extended to include both negative numbers and irrational numbers.
- The rule that one can calculate the value of an angle in a *r* harmonic chart by multiplying the angle in our original chart by *r* and reducing the result $modulo$ $360°$ was proven.
- The rule that one can easily find the 12 harmonic of a natal angle by simply multiplying its position within its sign by 12 (without the need to reduce the result $modulo$ $360°$) was proven.

- The rule that if *n* is a counting number, then there exist exactly *n* distinct *n* harmonic angles was proven.
- The rule that every positive rational number harmonic angle for a rational number $\frac{n}{m}$ in lowest terms, where *n* and *m* are counting numbers, is also a *n* harmonic angle was proven. Hence, the study of positive rational number harmonics can be reduced to the study of counting number harmonics.
- The rule that every positive rational number harmonic r with $0 < r < 1$ can be replaced by a positive rational number harmonic r' where $r' > 1$ was proven. The advantage of this is that the root r' harmonic angle will then automatically be equal to $\frac{360°}{r'}$.
- The rule that if *n* is a counting number greater than 1 and if $0° < \theta < 360°$, then θ is a *n* harmonic angle if and only $(n \cdot \theta)\ modulo\ 360° = 0°$ was proven.
- The rule that if *m* and *n* are counting numbers, then the *mn* harmonic is equal to the *m* harmonic of the *n* harmonic and is also equal to the *n* harmonic of the *m* harmonic was proven. This means that, for counting numbers, one can calculate the *mn* harmonic in stages. However, it was also shown that this relationship does not always hold if *m* and *n* are not counting numbers.
- The rule that 0° is a fixed point in the calculation of all harmonic charts was proven.
- The rule that *n* harmonic angles are fixed points when one converts a chart to its $n + 1$ harmonic was proven.
- The rule that *n* harmonic angles are fixed points when one converts a chart to a harmonic that is equivalent to a polynomial with integer coefficients and with constant term equal to 1 that has been evaluated at the value *n* was proven. This greatly generalizes the previous result.
- The rule that if θ is a *mn* harmonic angle, then $n \cdot \theta$ is a *m* harmonic angle was proven.
- The rule that if θ is the sum of a *n* harmonic and a *m* harmonic angle, then $n \cdot \theta$ is a *m* harmonic angle was proven.
- The rule that there is a duality between the *least common multiple* of two positive rational number harmonics and the *greatest common divisor* of their corresponding root harmonic angles was proven.
- The rule that the “wheel graph” of a counting number harmonic is often a regular polygon while the wheel graph of a positive rational number harmonic is often a star polygon was demonstrated visually without a formal proof.

The above results add rigorous mathematical structure to the field of harmonic astrology, and they provide a framework for the mathematical derivation of new results. However, this paper has been primarily quantitative in nature. What has not been addressed in this paper is a model for the qualitative interpretation of harmonics, and qualitative interpretation has been the primary focus of many harmonic and vibrational astrologers. That issue can be addressed, though, in another paper that would be complimentary to this one where interpretations derived from a model can be compared with interpretations that others have found empirically, and as has happened in many other disciplines, it is hoped that the interplay of theory and empirical observations will lead to even more rapid progress.

Acknowledgements

I would like to heartfully thank Robert Currey and Vincent Godbout for their valuable comments on drafts of this paper.

References

Addey, John, (2005), *Harmonic Anthology*, American Federation of Astrologers, ISBN 978-0866900614, First Printing 1976, Current Printing 2005

Benton, Christopher (2024), *Essential Theorems for Harmonic Astrology*, https://www.youtube.com/playlist?list=PLUR0MagCZR6e8dFBMuYG7NDgUMC_2rROd (YouTube playlist)

Cochrane, David, (2002), *Astrology for the 21st Century*, Cosmic Patterns Software Inc., ISBN 1-9716952-0-2, 1st edition

Cochrane, David & Berry, Linda, (2016), *Vibrational Astrology Interpreting Aspects*, , Cosmic Patterns Software Inc., ISBN 5-800118-793079

Cochrane, David, (2021), *Vibrational Astrology The Essentials*, Cosmic Patterns Software Inc., ISBN 978-0-9716952-2-1, 1st edition

Cochrane, David, (2012), *The First* 32 *Harmonics*, https://www.astrosoftware.com/harmonicfirst32.pdf, Cosmic Patterns Software Inc.

Cochrane, Fei & David, (2025) *Basic Alphabet of Astrology*, http://avalonastrology.com/cheatsheet.pdf, Retrieved: 17 February 2025

Mancuso, Santino (24 January 2013), *Introduction to Harmonic Astrology* https://www.youtube.com/watch?v=mhYpPZQhBFc (YouTube video)

Mancuso, Santino (6 Dec 2017), *Meanings of Harmonics* 1 *to* 8, https://www.youtube.com/watch?v=ZD1hDoBiOUE (YouTube video)

Mancuso, Santino (9 Dec 2017), *Meanings of Harmonics* 9 *to* 16, https://www.youtube.com/watch?v=5QawH0Fon9M (YouTube video)

Mancuso, Santino (15 Dec 2017), *Meanings of Harmonics* 17 *to* 24, https://www.youtube.com/watch?v=YSJNPzB7ckU (YouTube video)

Mancuso, Santino (18 Dec 2017), *Meanings of Harmonics* 25 *to* 32, https://www.youtube.com/watch?v=lEIHoXdEOQQ (YouTube video)

Mancuso, Santino (5 April 2018), *The Navamsa Chart: its Magic, Beauty, and Blessings*, https://www.youtube.com/watch?v=ALg-75UI6gc (YouTube video)

Mancuso, Santino (7 Nov 2018), *What is Vibrational Astrology?*, https://www.youtube.com/watch?v=ALg-75UI6gc (YouTube video)

---

[1] To find the *least common multiple* of two counting numbers, the following 3-step procedure can be used: (1) factor each counting number into a product of primes raised to the necessary powers, (2) select those primes raised to the highest powers shown, and (3) multiply all the selected primes raised to their respective powers together to get the *lcm*.

1. $12 = 2^2 \cdot 3$
   $30 = 2 \cdot 3 \cdot 5$

2. $12 = 2^2 \cdot 3$
   $30 = 2 \cdot 3 \cdot 5$

3. $lcm(12{,}30) = 2^2 \cdot 3 \cdot 5 = 60$

[2] To find the *greatest common divisor* of two counting numbers, the following 3-step procedure can be used: (1) factor each counting number into a product of primes raised to the necessary powers, (2) select those primes common to both factorizations raised to the lowest powers shown, and (3) multiply all the selected primes raised to their respective powers together to get the *gcd*.

1. $12 = 2^2 \cdot 3$
   $30 = 2 \cdot 3 \cdot 5$

2. $12 = 2^2 \cdot 3$
   $30 = 2 \cdot 3 \cdot 5$

3. $gcd(12,30) = 2 \cdot 3 = 6$

[3] A regular polygon is a closed geometric figure comprised of line segments of equal length that do not cross one another and such that the interior angles are all equal to one another. Classic examples of regular polygons are squares and equilateral triangles.