b. Show that for $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.

Definition: The Gamma Function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$.

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Prove: For $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.

Proof: By definition, $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$. Let $u = t^x$ and let $dv = e^{-t}\,dt$. Then, using Integration by Parts, we have $du = xt^{x-1}\,dt$, $v = -e^{-t}$, and $\int t^x e^{-t}\,dt = -t^x e^{-t} + \int xt^{x-1}e^{-t}\,dt = \frac{-t^x}{e^t} + x\int t^{x-1}e^{-t}\,dt$.

Hence, $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \left.\frac{-t^x}{e^t}\right|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x)$.