e. Compute $\Gamma\left(\frac{1}{2}\right)$.

By definition, $\Gamma\left(\frac{1}{2}\right)=\int_0^\infty t^{1/2-1}e^{-t}\,dt=\int_0^\infty t^{-1/2}e^{-t}\,dt$. If we make the substitution $t=u^2$, then $dt=2u\,du$. Also, the new limits of integration will still be from 0 to ∞. Hence,
$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty t^{-1/2}e^{-t}\,dt=\int_0^\infty (u^2)^{-1/2}e^{-u^2}\,2u\,du=2\int_0^\infty e^{-u^2}\,du$. Consequently,
$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=4\left(\int_0^\infty e^{-u^2}\,du\right)^2=4\cdot\int_0^\infty e^{-u^2}\,du\cdot\int_0^\infty e^{-u^2}\,du$. If we decide to use the label "*v*" for our variable in the second integral, then we can write this as
$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=4\cdot\int_0^\infty e^{-u^2}\,du\cdot\int_0^\infty e^{-v^2}\,dv=4\int_0^\infty\int_0^\infty e^{-u^2}e^{-v^2}\,dudv=4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,dudv$. We can now convert this integral to polar coordinates by letting $u=r\cos\theta$, $v=r\sin\theta$, and replacing $dudv$ with $rdrd\theta$. However, what will happen with our limits of integration? We might say that with respect to *u* and *v*, we are integrating over a rectangle defined by $0\le u\le m$ and $0\le v\le n$ where we ultimately let both *m* and *n* go to ∞. Also, we can convert (u,v) coordinates into (r,θ) coordinates by using the formulas $r=\sqrt{u^2+v^2}$ and $\theta=\tan^{-1}\left(\frac{v}{u}\right)$. We now ask ourselves what the range of values is for *r*. Clearly, $r=0$ when $u=0=v$, and on the rectangle defined by $0\le u\le m$ and $0\le v\le n$, then maximum value for *r* will be $\sqrt{m^2+n^2}$. Additionally, as both *m* and *n* go to infinity, *r* will also go to infinity. Therefore, the range for *r* is $0\le r<\infty$. (*r* will also take on every value between zero and infinity, as can easily be shown.) As for θ, the smallest value we have for $\tan^{-1}\left(\frac{v}{u}\right)$ is zero which will occur when $v=0$ and $u\ne 0$. Also, for any fixed value of *v*, as we let $u\to 0$, we will have $\frac{v}{u}\to\infty$. Hence, $\theta\to\frac{\pi}{2}$ as $\frac{v}{u}\to\infty$, and the range of values for θ is $0\le\theta<\frac{\pi}{2}$. Notice, too, that we could integrate over the range $0\le\theta\le\frac{\pi}{2}$ without altering the value of the integral. Thus,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,dudv=4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,rdrd\theta=-2\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,(-2r)drd\theta=-2\int_0^{\pi/2} e^{-r^2}\Big|_0^\infty\,d\theta$$

$$=\lim_{n\to\infty}\left(-2\int_0^{\pi/2}\left[\frac{1}{e^{n^2}}-\frac{1}{e^0}\right]\right)d\theta=2\int_0^{\pi/2}d\theta=2\theta\Big|_0^{\pi/2}=2\left(\frac{\pi}{2}\right)-2\cdot 0=\pi.$$ Therefore, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.