g. Show, using definitions, that $L[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$.

Clearly, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt \Rightarrow \Gamma(\alpha+1) = \int_0^{\infty} t^{\alpha} e^{-t}\, dt$. Also,

$L[f(x)] = \int_0^{\infty} e^{-sx} f(x)\, dx \Rightarrow L[t^{\alpha}] = \int_0^{\infty} e^{-st} t^{\alpha}\, dt$. Let $u = st \Rightarrow t = \frac{u}{s}$ and $du = s\, dt \Rightarrow dt = \frac{1}{s} du$.

Notice that with this substitution our limits of integration are still from zero to infinity.

Thus, $L[t^{\alpha}] = \int_0^{\infty} e^{-st} t^{\alpha}\, dt = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{1}{s} du = \frac{1}{s^{\alpha+1}} \int_0^{\infty} u^{\alpha} e^{-u}\, du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$.