THE LAPLACE TRANSFORM

$$L\left[f(x) \right] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

$$L[1]=\frac{1}{s},\quad (s>0)$$

$$L\left[e^{rx}\right]=\frac{1}{s-r},\quad (s>r)$$

$$L[\sin ax]=\frac{a}{s^2+a^2},\quad (s>0)$$

$$L[\cos ax]=\frac{s}{s^2+a^2},\quad (s>0)$$

$$L[x]=\frac{1}{s^2},\quad (s>0)$$

$$L\left[x^2\right]=\frac{2}{s^3},\quad (s>0)$$

$$L\left[x^n\right]=\frac{n!}{s^{n+1}},\quad (s>0)$$

$$L\left[xe^{rx}\right]=\frac{1}{(s-r)^2},\quad (s>r)$$

$$L\left[x^2e^{rx}\right]=\frac{2}{(s-r)^3},\quad (s>r)$$

$$L\left[x^ne^{rx}\right]=\frac{n!}{(s-r)^{n+1}},\quad (s>0)$$

$$L[x\sin ax]=\frac{2as}{(s^2+a^2)^2},\quad (s>0)$$

$$L[x\cos ax]=\frac{s^2-a^2}{(s^2+a^2)^2},\quad (s>0)$$

$$L\left[e^{ax}\sin bx\right]=\frac{b}{(s-a)^2+b^2},\ (s>a)$$

$$L\left[e^{ax}\cos bx\right]=\frac{s-a}{(s-a)^2+b^2},\ (s>a)$$

$$L\left[c\cdot f(x)\right]=c\cdot L\left[f(x)\right]$$

$$L\left[c\right]=\frac{c}{s}$$

$$L\left[f(x)+g(x)\right]=L\left[f(x)\right]+L\left[g(x)\right]$$

$$L\left[x^2\right]=\frac{2}{s^3},\quad (s>0)$$

Example 1: $f(x) = 1$

$$L[1] = \int_0^\infty e^{-sx} \cdot 1 dx = \int_0^\infty e^{-sx} dx = \left.\frac{e^{-sx}}{-s}\right|_0^\infty = \left.\frac{1}{-se^{sx}}\right|_0^\infty$$

$$= \lim_{t\to\infty}\left[\frac{1}{-se^{st}} - \frac{1}{-se^{0}}\right] = \frac{1}{s}, \qquad (\text{if } s > 0)$$

Example 2: $f(x) = e^{rx}$

$$L\left[e^{rx}\right] = \int_0^\infty e^{-sx} e^{rx}\, dx = \int_0^\infty e^{rx-sx}\, dx = \int_0^\infty e^{(r-s)x}\, dx = \left.\frac{e^{(r-s)x}}{r-s}\right|_0^\infty$$

$$= -\left.\frac{1}{(s-r)e^{(s-r)x}}\right|_0^\infty = \lim_{t\to\infty}\left[-\frac{1}{(s-r)e^{(s-r)t}} + \frac{1}{(s-r)e^0}\right]$$

$$= \frac{1}{s-r}, \qquad (\text{if } s > r)$$

Example 3: $f(x) = \sin(ax)$

$$L\left[\sin ax\right] = \int_0^\infty e^{-sx}\sin(ax)\,dx = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = e^{-sx}$$

$$dv = \sin(ax)\,dx$$

$$v = \frac{-\cos(ax)}{a} \qquad = e^{-sx}\cdot\frac{-\cos(ax)}{a}\Big|_0^\infty - \int_0^\infty -se^{-sx}\cdot\frac{-\cos(ax)}{a}\,dx$$

$$du = -se^{-sx}\,dx$$

$$= \lim_{t\to\infty}\left(\frac{1}{e^{st}}\cdot\frac{-\cos(at)}{a} + \frac{1}{a}\right) - \frac{s}{a}\int_0^\infty e^{-sx}\cos(ax)\,dx$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-sx}\cos(ax)\,dx \quad (s > 0)$$

$$L[\sin ax] = \int_0^\infty e^{-sx}\sin(ax)\,dx = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-sx}\cos(ax)\,dx$$

$$u = e^{-sx}$$

$$dv = \cos(ax)\,dx$$

$$v = \frac{\sin(ax)}{a}$$

$$= \frac{1}{a} - \frac{s}{a}\left(e^{-sx}\cdot\frac{\sin(ax)}{a}\Bigg|_0^\infty + \frac{s}{a}\int_0^\infty e^{-sx}\sin(ax)\,dx \right)$$

$$du = -se^{-sx}\,dx$$

$$= \frac{1}{a} - \frac{s}{a}\left[\lim_{t\to\infty}\left(\frac{1}{e^{st}}\cdot\frac{\sin(at)}{a} - 0 \right) + \frac{s}{a}\int_0^\infty e^{-sx}\sin(ax)\,dx \right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-sx}\sin(ax)\,dx \quad (s>0)$$

$$L[\sin ax] = \int_0^{\infty} e^{-sx} \sin(ax)\,dx = \frac{1}{a} - \frac{s^2}{a^2}\int_0^{\infty} e^{-sx} \sin(ax)\,dx \quad (s>0)$$

$$\Rightarrow \frac{s^2}{a^2}\int_0^{\infty} e^{-sx} \sin(ax)\,dx + \int_0^{\infty} e^{-sx} \sin(ax)\,dx$$

$$= \left(\frac{s^2}{a^2}+1\right)\int_0^{\infty} e^{-sx} \sin(ax)\,dx = \left(\frac{s^2+a^2}{a^2}\right)\int_0^{\infty} e^{-sx} \sin(ax)\,dx = \frac{1}{a}$$

$$\Rightarrow \int_0^{\infty} e^{-sx} \sin(ax)\,dx = \frac{a^2}{s^2+a^2}\cdot\frac{1}{a} = \frac{a}{s^2+a^2}, \quad (s>0)$$

Example 4: $f(x) = \cos(ax)$

$$L[\cos ax] = \int_0^\infty e^{-sx}\cos(ax)\,dx = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = e^{-sx}$$

$$dv = \cos(ax)\,dx$$

$$v = \frac{\sin(ax)}{a}$$

$$= e^{-sx}\cdot\frac{\sin(ax)}{a}\Bigg|_0^\infty - \int_0^\infty -se^{-sx}\cdot\frac{\sin(ax)}{a}\,dx$$

$$du = -se^{-sx}\,dx$$

$$= \lim_{t\to\infty}\left(\frac{1}{e^{st}}\cdot\frac{\sin(at)}{a} - 0\right) + \frac{s}{a}\int_0^\infty e^{-sx}\sin(ax)\,dx$$

$$= \frac{s}{a}\int_0^\infty e^{-sx}\sin(ax)\,dx \quad (s > 0)$$

$$L[\cos ax] = \int_0^\infty e^{-sx}\cos(ax)\,dx = \frac{s}{a}\int_0^\infty e^{-sx}\sin(ax)\,dx$$

$$u = e^{-sx}$$

$$dv = \sin(ax)\,dx$$

$$v = \frac{-\cos(ax)}{a}$$

$$= \frac{s}{a}\left(e^{-sx}\cdot\frac{-\cos(ax)}{a}\Bigg|_0^\infty - \frac{s}{a}\int_0^\infty e^{-sx}\cos(ax)\,dx \right)$$

$$du = -se^{-sx}\,dx$$

$$= \frac{s}{a}\left[\lim_{t\to\infty}\left(\frac{1}{e^{st}}\cdot\frac{-\cos(at)}{a} + \frac{1}{a}\right) - \frac{s}{a}\int_0^\infty e^{-sx}\cos(ax)\,dx\right]$$

$$= \frac{s}{a^2} - \frac{s^2}{a^2}\int_0^\infty e^{-sx}\cos(ax)\,dx \quad (s > 0)$$

$$L[\cos ax] = \int_0^\infty e^{-sx}\cos(ax)\,dx = \frac{s}{a^2} - \frac{s^2}{a^2}\int_0^\infty e^{-sx}\cos(ax)\,dx \quad (s>0)$$

$$\Rightarrow \frac{s^2}{a^2}\int_0^\infty e^{-sx}\cos(ax)\,dx + \int_0^\infty e^{-sx}\cos(ax)\,dx$$

$$= \left(\frac{s^2}{a^2}+1\right)\int_0^\infty e^{-sx}\cos(ax)\,dx = \left(\frac{s^2+a^2}{a^2}\right)\int_0^\infty e^{-sx}\cos(ax)\,dx = \frac{s}{a^2}$$

$$\Rightarrow \int_0^\infty e^{-sx}\cos(ax)\,dx = \frac{a^2}{s^2+a^2}\cdot\frac{s}{a^2} = \frac{s}{s^2+a^2}, \quad (s>0)$$

Example 5: $f(x) = x$

$$L[x]=\int_0^\infty xe^{-sx}\,dx = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = x$$

$$dv = e^{-sx}\,dx$$

$$v = \frac{e^{-sx}}{-s} \qquad = \frac{xe^{-sx}}{-s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-sx}\,dx$$

$$du = dx$$

$$= \lim_{t\to\infty}\left(\frac{x}{-se^{st}} + 0\right) + \frac{1}{s}\cdot\frac{e^{-sx}}{-s}\Bigg|_0^\infty$$

$$= \lim_{t\to\infty}\left[\frac{1}{s}\left(\frac{1}{-se^{st}} + \frac{1}{s}\right)\right] = \frac{1}{s^2}, \quad (s>0)$$

Example 6: $f(x) = x^2$

$$L\left[x^2\right] = \int_0^\infty x^2 e^{-sx}\,dx = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = x^2$$

$$dv = e^{-sx}\,dx$$

$$v = \frac{e^{-sx}}{-s}$$

$$= \frac{x^2 e^{-sx}}{-s}\Bigg|_0^\infty + \frac{2}{s}\int_0^\infty x e^{-sx}\,dx$$

$$du = 2x\,dx$$

$$= \lim_{t\to\infty}\left(\frac{x^2}{-se^{st}} + 0\right) + \frac{2}{s}\cdot\frac{1}{s^2}$$

$$= \frac{2}{s^3}, \quad (s > 0)$$

Example 7: $f(x) = x^n$

Theorem: $L\left[x^n\right]=\dfrac{n!}{s^{n+1}}$ for all counting numbers n and $s>0$.

Proof: By previous proof, $L[x]=\dfrac{1}{s^2}$. Now assume it is true that for some n,

$L\left[x^n\right]=\dfrac{n!}{s^{n+1}}$. Then we want to show that $L\left[x^{n+1}\right]=\dfrac{(n+1)!}{s^{n+2}}$. But clearly,

$L\left[x^{n+1}\right]=\int_0^\infty x^{n+1}e^{-sx}\,dx$. Let $u=x^{n+1}, dv=e^{-sx}dx, v=\dfrac{e^{-sx}}{-s}, du=(n+1)x^n dx$.

Then $L\left[x^{n+1}\right]=\int_0^\infty x^{n+1}e^{-sx}\,dx=\left.\dfrac{x^{n+1}e^{-sx}}{-s}\right|_0^\infty+\dfrac{n+1}{s}\int_0^\infty x^n e^{-sx}\,dx$

$=0+\dfrac{n+1}{s}\cdot\dfrac{n!}{s^{n+1}}=\dfrac{(n+1)!}{s^{n+2}}$ $(s>0)$. Therefore, by mathematical induction,

$L\left[x^n\right]=\dfrac{n!}{s^{n+1}}$ for all counting numbers n and $s>0$. □

Example 8: $f(x) = xe^{rx}$

$$L\left[xe^{rx}\right]=\int_0^\infty xe^{rx}e^{-sx}\,dx=\int_0^\infty xe^{(r-s)x}\,dx=uv\Big|_0^\infty-\int_0^\infty v\,du$$

$$u=x$$

$$dv=e^{(r-s)x}\,dx$$

$$v=\frac{e^{(r-s)x}}{r-s}$$

$$=\frac{xe^{(r-s)x}}{r-s}\Bigg|_0^\infty-\frac{1}{r-s}\int_0^\infty e^{(r-s)x}\,dx$$

$$du=dx$$

$$=\lim_{t\to\infty}\left(\frac{t}{(r-s)e^{(s-r)t}}-0\right)-\frac{1}{r-s}\left(\frac{e^{(r-s)x}}{r-s}\right)\Bigg|_0^\infty$$

$$=-\frac{1}{r-s}\cdot\lim_{t\to\infty}\left(\frac{1}{(r-s)e^{(s-r)t}}-\frac{1}{r-s}\right)=\frac{1}{(r-s)^2}=\frac{1}{(s-r)^2},\quad (s>r)$$

Example 9: $f(x) = x^2 e^{rx}$

$$L\left[x^2e^{rx}\right]=\int_0^\infty x^2e^{rx}e^{-sx}\,dx=\int_0^\infty x^2e^{(r-s)x}\,dx=uv\Big|_0^\infty-\int_0^\infty v\,du$$

$$u=x^2$$

$$dv=e^{(r-s)x}\,dx$$

$$v=\frac{e^{(r-s)x}}{r-s}$$

$$=\frac{2xe^{(r-s)x}}{r-s}\Bigg|_0^\infty-\frac{2}{r-s}\int_0^\infty xe^{(r-s)x}\,dx$$

$$du=2x\,dx$$

$$=\lim_{t\to\infty}\left(\frac{2t}{(r-s)e^{(s-r)t}}-0\right)-\frac{2}{r-s}\cdot\frac{1}{(s-r)^2}$$

$$=\frac{2}{s-r}\cdot\frac{1}{(s-r)^2}=\frac{2}{(s-r)^3},\quad (s>r)$$

Example 10: $f(x) = x^n e^{rx}$

Theorem: $L\left[x^n e^{rx}\right] = \frac{n!}{(s-r)^{n+1}}$ for all counting numbers n and $s > 0$.

Proof: By previous proof, $L\left[xe^{rx}\right] = \frac{1}{(s-r)^2}$. Now assume it is true that for some n,

$L\left[x^n e^{rx}\right] = \frac{n!}{(s-r)^{n+1}}$. Then we want to show that $L\left[x^{n+1} e^{rx}\right] = \frac{(n+1)!}{(s-r)^{n+2}}$. But clearly,

$L\left[x^{n+1} e^{rx}\right] = \int_0^\infty x^{n+1} e^{(r-s)x}\,dx$. Let $u = x^{n+1}, dv = e^{(r-s)x}dx, v = \frac{e^{(r-s)x}}{r-s}, du = (n+1)x^n dx.$

Then $L\left[x^{n+1} e^{rx}\right] = \int_0^\infty x^{n+1} e^{(r-s)x}\,dx = \left.\frac{x^{n+1} e^{(r-s)x}}{r-s}\right|_0^\infty - \frac{n+1}{r-s}\int_0^\infty x^n e^{(r-s)x}\,dx$

$= 0 + \frac{n+1}{s-r} \cdot \frac{n!}{(s-r)^{n+1}} = \frac{(n+1)!}{(s-r)^{n+2}}$ $(s > r)$. Therefore, by mathematical induction,

$L\left[x^n e^{rx}\right] = \frac{n!}{(s-r)^{n+1}}$ for all counting numbers n and $s > r$. □

Example 11: $f(x) = x \sin ax$

$$L\left[x\sin ax\right]=\int_0^\infty x\sin(ax)e^{-sx}\,dx,\qquad (s>0)$$

$$u=x\sin ax$$

$$dv=e^{-sx}\,dx$$

$$v=\frac{e^{-sx}}{-s}$$

$$du=(xa\cos ax+\sin ax)\,dx$$

$$=\left.\frac{x\sin(ax)e^{-sx}}{-s}\right|_0^\infty-\int_0^\infty\frac{e^{-sx}}{-s}(xa\cos ax+\sin ax)\,dx$$

$$=\frac{a}{s}\int_0^\infty x\cos(ax)e^{-sx}\,dx+\frac{1}{s}\int_0^\infty\sin(ax)e^{-sx}\,dx$$

$$= \frac{a}{s}\int_0^\infty x\cos(ax)e^{-sx}\,dx + \frac{1}{s}\cdot\frac{a}{s^2+a^2}$$

$$u = x\cos ax$$

$$dv = e^{-sx}\,dx$$

$$v = \frac{e^{-sx}}{-s}$$

$$du = (-xa\sin ax + \cos ax)\,dx$$

$$= \frac{a}{s}\left[\left.\frac{x\cos(ax)e^{-sx}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s}(-xa\sin ax + \cos ax)\,dx\right] + \frac{a}{s(s^2+a^2)}$$

$$= \frac{a}{s}\left[\frac{-a}{s}\int_0^\infty e^{-sx}x\sin ax\,dx + \frac{1}{s}\int_0^\infty e^{-sx}\cos ax\,dx\right] + \frac{a}{s(s^2+a^2)}$$

$$= \frac{-a^2}{s^2}\int_0^\infty e^{-sx} x \sin ax\, dx + \frac{a}{s^2}\cdot\frac{s}{s^2+a^2} + \frac{a}{s(s^2+a^2)}$$

$$\Rightarrow \frac{s^2+a^2}{s^2}\int_0^\infty e^{-sx} x \sin ax\, dx = \frac{a}{s(s^2+a^2)} + \frac{a}{s(s^2+a^2)} = \frac{2a}{s(s^2+a^2)}$$

$$\Rightarrow L\left[x \sin ax\right] = \int_0^\infty e^{-sx} x \sin ax\, dx = \frac{2as}{(s^2+a^2)^2}, \qquad (s>0)$$

Example 12: $f(x) = x\cos ax$

$$L\left[x\cos ax\right]=\int_0^\infty x\cos(ax)e^{-sx}\,dx, \qquad (s>0)$$

$$u = x\cos ax$$

$$dv = e^{-sx}\,dx$$

$$v = \frac{e^{-sx}}{-s}$$

$$du = (-xa\sin ax + \cos ax)\,dx$$

$$= \left.\frac{x\cos(ax)e^{-sx}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s}(-xa\sin ax + \cos ax)\,dx$$

$$= -\frac{a}{s}\int_0^\infty x\sin(ax)e^{-sx}\,dx + \frac{1}{s}\int_0^\infty \cos(ax)e^{-sx}\,dx$$

$$= -\frac{a}{s}\int_0^\infty x\sin(ax)e^{-sx}\,dx + \frac{1}{s}\cdot\frac{s}{s^2+a^2}$$

$$u = x\sin ax$$

$$dv = e^{-sx}\,dx$$

$$v = \frac{e^{-sx}}{-s}$$

$$du = (xa\cos ax + \sin ax)\,dx$$

$$= -\frac{a}{s}\left[\left.\frac{x\sin(ax)e^{-sx}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s}(xa\cos ax + \sin ax)\,dx\right] + \frac{1}{s^2+a^2}$$

$$= -\frac{a}{s}\left[\frac{a}{s}\int_0^\infty e^{-sx}x\cos ax\,dx + \frac{1}{s}\int_0^\infty e^{-sx}\sin ax\,dx\right] + \frac{1}{s^2+a^2}$$

$$= \frac{-a^2}{s^2}\int_0^{\infty} e^{-sx} x\cos ax\,dx - \frac{a}{s^2}\cdot\frac{a}{s^2+a^2} + \frac{1}{s^2+a^2}$$

$$\Rightarrow \frac{s^2+a^2}{s^2}\int_0^{\infty} e^{-sx} x\cos ax\,dx = \frac{-a^2}{s^2(s^2+a^2)} + \frac{1}{s^2+a^2}$$

$$\Rightarrow L\left[x\cos ax\right] = \int_0^{\infty} e^{-sx} x\cos ax\,dx = \frac{s^2-a^2}{(s^2+a^2)^2}, \qquad (s>0)$$

Example 13: $f(x) = e^{ax} \sin bx$

$$L\left[e^{ax}\sin bx\right] = \int_0^\infty e^{-sx}e^{ax}\sin(bx)\,dx = \int_0^\infty e^{(a-s)x}\sin(bx)\,dx, \qquad (s > a)$$

$$u = \sin bx$$

$$dv = e^{(a-s)x}\,dx$$

$$v = \frac{e^{(a-s)x}}{a-s}$$

$$du = b\cos(bx)\,dx$$

$$= \left.\frac{e^{(a-s)x}\sin bx}{a-s}\right|_0^\infty - \int_0^\infty \frac{e^{(a-s)x}}{a-s} b\cos(bx)\,dx$$

$$= -\frac{b}{a-s}\int_0^\infty e^{(a-s)x}\cos(bx)\,dx$$

$$= -\frac{b}{a-s}\int_0^{\infty} e^{(a-s)x}\cos(bx)\,dx$$

$$u = \cos bx$$

$$dv = e^{(a-s)x}\,dx$$

$$v = \frac{e^{(a-s)x}}{a-s}$$

$$du = -b\sin(bx)\,dx$$

$$= -\frac{b}{a-s}\left[\left.\frac{e^{(a-s)x}\cos bx}{a-s}\right|_0^{\infty} + \frac{b}{a-s}\int_0^{\infty} e^{(a-s)x}\sin(bx)\,dx\right]$$

$$= \frac{b}{(a-s)^2} - \frac{b^2}{(a-s)^2}\int_0^{\infty} e^{(a-s)x}\sin(bx)\,dx$$

$$\Rightarrow \frac{(a-s)^2+b^2}{(a-s)^2}\int_0^\infty e^{(a-s)x}\sin(bx)\,dx = \frac{b}{(a-s)^2}$$

$$\Rightarrow \int_0^\infty e^{(a-s)x}\sin(bx)\,dx = \frac{b}{(a-s)^2+b^2}$$

$$\Rightarrow L\left[e^{ax}\sin bx\right] = \frac{b}{(s-a)^2+b^2}, \qquad (s>a)$$

Example 14: $f(x) = e^{ax} \cos bx$

$$L\left[e^{ax}\cos bx\right]=\int_0^\infty e^{-sx}e^{ax}\cos(bx)\,dx=\int_0^\infty e^{(a-s)x}\cos(bx)\,dx,\qquad (s>a)$$

$$u=\cos bx$$

$$dv=e^{(a-s)x}\,dx$$

$$v=\frac{e^{(a-s)x}}{a-s}$$

$$du=-b\sin(bx)\,dx$$

$$=\left.\frac{e^{(a-s)x}\cos bx}{a-s}\right|_0^\infty+\int_0^\infty\frac{e^{(a-s)x}}{a-s}b\sin(bx)\,dx$$

$$=\frac{-1}{a-s}+\frac{b}{a-s}\int_0^\infty e^{(a-s)x}\sin(bx)\,dx$$

$$= \frac{-1}{a-s} + \frac{b}{a-s}\int_0^\infty e^{(a-s)x}\sin(bx)\,dx$$

$$u = \sin bx$$

$$dv = e^{(a-s)x}\,dx$$

$$v = \frac{e^{(a-s)x}}{a-s}$$

$$du = b\cos(bx)\,dx$$

$$= \frac{-1}{a-s} + \frac{b}{a-s}\left[\left.\frac{e^{(a-s)x}\sin bx}{a-s}\right|_0^\infty - \frac{b}{a-s}\int_0^\infty e^{(a-s)x}\cos(bx)\,dx\right]$$

$$= \frac{-1}{a-s} - \frac{b^2}{(a-s)^2}\int_0^\infty e^{(a-s)x}\cos(bx)\,dx$$

$$\Rightarrow \frac{(a-s)^2+b^2}{(a-s)^2}\int_0^{\infty} e^{(a-s)x}\cos(bx)\,dx = \frac{s-a}{(a-s)^2}$$

$$\Rightarrow \int_0^{\infty} e^{(a-s)x}\cos(bx)\,dx = \frac{s-a}{(a-s)^2+b^2}$$

$$\Rightarrow L\left[e^{ax}\cos bx\right] = \frac{s-a}{(s-a)^2+b^2}, \qquad (s>a)$$

Theorem: $L[c \cdot f(x)] = c \cdot L[f(x)]$.

Proof: Clearly, $L[c \cdot f(x)] = \int_0^\infty e^{-sx} c \cdot f(x)\, dx = \mathrm{c} \cdot \int_0^\infty e^{-sx} f(x)\, dx = c \cdot L[f(x)]$.

Therefore, $L[c \cdot f(x)] = c \cdot L[f(x)]$. □

Corollary: $L[c] = \frac{c}{s}$.

Proof: Clearly, $L[c] = L[c \cdot 1] = c \cdot L[1] = c \cdot \frac{1}{s} = \frac{c}{s}$.

Therefore, $L[c] = \frac{c}{s}$. □

Theorem: $L\left[f(x)+g(x)\right]=L\left[f(x)\right]+L\left[g(x)\right].$

Proof: Clearly, $L\left[f(x)+g(x)\right]=\int_0^\infty e^{-sx}[f(x)+g(x)]\,dx=$

$$=\int_0^\infty\left(e^{-sx}f(x)+e^{-sx}g(x)\right)dx=\int_0^\infty e^{-sx}f(x)\,dx+\int_0^\infty e^{-sx}g(x)\,dx=L\left[f(x)\right]+\mathrm{L}\left[g(x)\right].$$

Therefore, $L\left[f(x)+g(x)\right]=L\left[f(x)\right]+L\left[g(x)\right].$ $\square$

Example: $L\left[\sin ax+\cos ax\right]=L\left[\sin ax\right]+L\left[\cos ax\right]$

$$=\frac{a}{s^2+a^2}+\frac{s}{s^2+a^2}=\frac{s+a}{s^2+a^2},\qquad (s>0)$$