Change of Variables
In
Multiple Integrals
When we convert a double integral from rectangular to polar coordinates, recall the changes that must be made to $x$, $y$ and $dA$.

\[ x = x(r, \theta) = r \cos \theta \]
\[ y = y(r, \theta) = r \sin \theta \]
\[ dA = r \, dr \, d\theta \]
In the polar coordinate system, an element of area is generally a rectangle corresponding to a range of values for $r$ and a range for $\theta$. 

\[ a \leq r \leq b \quad \text{and} \quad c \leq \theta \leq d \]
However, in the $xy$ coordinate system, this rectangle usually takes on a different shape, and the formula for an element of area changes.

$$\Delta A \approx r \cdot \Delta r \cdot \Delta \theta$$
This leads to the following formula for the double integral in polar coordinates.

\[ \int \int_R f(x, y) \, dA = \int \int_T f(r \cos \theta, r \sin \theta) \, rdrd\theta \]
We’ll now develop a general method for finding change of variable formulas such as this one.

\[ \int \int_R f(x, y) \, dA = \int \int_T f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \]
Suppose we have a rectangle in an $st$-coordinate system and a pair of functions that converts $(s,t)$ coordinates into $(x,y)$ coordinates.

\[
x = x(s,t)
\]

\[
y = y(s,t)
\]
Suppose also that these functions are differentiable and that the transformation from the $st$-coordinates to $xy$-coordinates is one-to-one.

\[ x = x(s, t) \]

\[ y = y(s, t) \]
Then because of differentiability, local linearity will be present and a small rectangle in the \( st \) system will be mapped onto approximately a parallelogram in the \( xy \)-coordinate system.

\[
\begin{align*}
T \\
x &= x(s, t) \\
y &= y(s, t)
\end{align*}
\]
If we add some coordinates, then it looks like this.

\[ x = x(s, t) \]
\[ y = y(s, t) \]
An element of area in our $xy$-coordinate system is represented by a parallelogram.

\[ x = x(s,t) \]
\[ y = y(s,t) \]
The area of this parallelogram is given by the norm of a cross product.

\[ \text{Area} = \| \vec{a} \times \vec{b} \| \]
Notice that,

\[ \vec{a} = (x(s + \Delta s, t) - x(s, t))\hat{i} + (y(s + \Delta s, t) - y(s, t))\hat{j} \]
\[ \approx \frac{\partial x}{\partial s} \Delta s \hat{i} + \frac{\partial y}{\partial s} \Delta s \hat{j} \]

\[ \vec{b} = (x(s, t + \Delta t) - x(s, t))\hat{i} + (y(s, t + \Delta t) - y(s, t))\hat{j} \]
\[ \approx \frac{\partial x}{\partial t} \Delta t \hat{i} + \frac{\partial y}{\partial t} \Delta t \hat{j} \]
Hence,

\[
\vec{a} \times \vec{b} \approx \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \Delta s \\
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \Delta t \\
\end{vmatrix} = \left( \frac{\partial x}{\partial s} \Delta s \cdot \frac{\partial y}{\partial t} \Delta t - \frac{\partial y}{\partial s} \Delta s \cdot \frac{\partial x}{\partial t} \Delta t \right) \hat{k}
\]
And,

\[ \| \vec{a} \times \vec{b} \| \approx \left| \frac{\partial x}{\partial s} \Delta s \cdot \frac{\partial y}{\partial t} \Delta t - \frac{\partial y}{\partial s} \Delta s \cdot \frac{\partial x}{\partial t} \Delta t \right| = \left| \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} \right| \Delta s \Delta t \]
The expression inside the last absolute value sign is called the *Jacobian*, and it is usually written as follows:

\[
\frac{\partial (x, y)}{\partial (s, t)} = \begin{vmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}
\end{vmatrix} = \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t}
\]
Therefore,

\[
Area = \Delta A = \| \vec{a} \times \vec{b} \| \approx \left| \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} \right| \Delta s \Delta t = \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \Delta s \Delta t
\]

\[
(x(s,t), y(s,t)) \quad \quad (x(s,t+\Delta t), y(s,t+\Delta t)) \quad \quad (x(s+\Delta s,t), y(s+\Delta s,t))
\]

\[
\vec{a} \quad \vec{b} \quad R
\]
And as a result of local linearity, this approximation improves as the changes in $s$ and $t$ become smaller.

\[
Area = \Delta A = \| \vec{a} \times \vec{b} \| \approx \left| \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} \right| \Delta s \Delta t = \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \Delta s \Delta t
\]
And finally,

\[
\iint_{R} f(x, y) \, dA = \lim_{\Delta A \to 0} \sum f(x, y) \cdot \Delta A
\]

\[
= \lim_{\Delta s, \Delta t \to 0} \sum f(x(s, t), y(s, t)) \cdot \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \Delta s \Delta t
\]

\[
= \iint_{T} f(x(s, t), y(s, t)) \cdot \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \, ds \, dt
\]
Now let’s verify that this formula works for polar coordinates.

\[
\iint_R f(x, y) \, dA = \lim_{\Delta A \to 0} \sum f(x, y) \cdot \Delta A
\]

\[
= \lim_{\Delta s, \Delta t \to 0} \sum f(x, y) \cdot \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \Delta s \Delta t = \iint_T f(x, y) \cdot \left| \frac{\partial (x, y)}{\partial (s, t)} \right| \, ds \, dt
\]
\[
\int \int_{R} f(x, y) \, dA = \int \int_{T} f(x(s, t), y(s, t)) \cdot \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta
\]

\[
x = r \cos \theta \\
y = r \sin \theta
\]

\[
\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r \left( \cos^2 \theta + \sin^2 \theta \right) = r
\]

\[
\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \left| r \right| = r
\]
Therefore,

\[
\iint_{R} f(x, y) \, dA = \iint_{T} f(r \cos \theta, r \sin \theta) \cdot \left| \frac{\partial (x, y)}{\partial (r, \theta)} \right| \, dr \, d\theta
\]

\[
= \iint_{T} f(r \cos \theta, r \sin \theta) \cdot r \, dr \, d\theta
\]
Let’s try one more example! Consider the equation below for an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
The following substitution will change it into an equation for a unit circle.

\[ x = a \cdot s \]
\[ y = b \cdot t \]

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{(a \cdot s)^2}{a^2} + \frac{(b \cdot t)^2}{b^2} = 1 \Rightarrow s^2 + t^2 = 1 \]
Now find the absolute value of the Jacobian.

\[ x = a \cdot s \]
\[ y = b \cdot t \]

\[ \frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \]

\[ \left| \frac{\partial(x, y)}{\partial(s, t)} \right| = ab \]
We can now easily find the area of the ellipse.

\[
\text{Area of ellipse} = \iint_{\text{ellipse}} dA = \iint_{\text{unit circle}} ab \, dsdt
\]

\[
= ab \left( \iint_{\text{unit circle}} dsdt \right) = ab \cdot \pi = \pi ab
\]
If we have a function of three variables, then our Jacobian looks like this.

\[
\begin{vmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\
\frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u}
\end{vmatrix}
\]
Find the volume of the ellipsoid.

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]
The coordinate change below transforms the ellipsoid into a unit sphere.

\[
x = a \cdot s \\
y = b \cdot t \\
z = c \cdot u
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

\[\text{Volume} = \iiint_R dV = \iiint_T \left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| dsdtdudv\]

\[= \iiint_T abc dsdtdudv = abc \iiint_T dsdtdudv\]

\[= abc \cdot \frac{4\pi}{3} = \frac{4}{3} \pi abc\]
Any Questions?