STOKES’ THEOREM IN HIGHER DIMENSIONS
If we apply Stokes’ Theorem to a vector field $F=\langle P, Q \rangle$ and a plane curve $C_R$ that is oriented counterclockwise and that bounds a region $R$, then we get the following formula (the same as Green’s Theorem):

$$\int_{C_R} F \cdot dr = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \text{curl } F \cdot \hat{k} dA = \iint_R (\nabla \times F) \cdot \hat{k} dA$$
Notice in this formula that we are integrating the dot product of the curl of $F$ with an upward pointing unit normal vector $k$.

\[
\int_{C_R} F \cdot dr = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \text{curl } F \cdot \hat{k} \, dA = \iint_R (\nabla \times F) \cdot \hat{k} \, dA
\]
Thus, it should come as no surprise that if we integrate around a counterclockwise oriented curve $C$ that bounds a surface $S$, then our formula will involve both an upward pointing unit normal and a surface integral.
Let’s suppose that our surface is the graph of $z=f(x,y)$. Then consider this as a level surface for the function $g(x,y,z)=z-f(x,y)$, and define an upward pointing unit normal as follows:

$$N = \frac{\nabla g}{||\nabla g||} = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$
Notice that our unit normal will point upward because the $k$ component is positive.

\[
N = \frac{\nabla g}{\|\nabla g\|} = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}
\]
We can now write out the higher dimensional version of Stokes’ Theorem.

\[ \int_C F \cdot dr = \iint_S (\nabla \times F) \cdot N \, dS = \iint_S (\nabla \times F) \cdot \frac{\nabla g}{\|\nabla g\|} \, dS \]

\[ = \iint_R (\nabla \times F) \cdot \frac{\nabla g}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1} \, dA = \iint_R (\nabla \times F) \cdot \nabla g \, dA \]
Now let’s do a problem!

\[ S: z = -x^4 - y^4 \]
\[ R: 0 \leq x \leq 1, 0 \leq y \leq 1 \]
\[ F = z \mathbf{i} + x \mathbf{j} + y \mathbf{k} \]

\[ \int_C F \cdot dr = \int_R (\nabla \times F) \cdot \nabla g \, dA \]
First, find the gradient of $g$ and the curl of $F$.

$S: z = -x^4 - y^4$

$R: 0 \leq x \leq 1, \ 0 \leq y \leq 1$

$F = z \hat{i} + x \hat{j} + y \hat{k}$

\[
\int_C F \cdot dr = \iint_R (\nabla \times F) \cdot \nabla g \, dA
\]

$g = x^4 + y^4 + z$

\[
\nabla g = 4x^3 \hat{i} + 4y^3 \hat{j} + \hat{k}
\]

\[
\nabla \times F = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z & x & y
\end{vmatrix} = \hat{i} + \hat{j} + \hat{k}
\]
And now, integrate!

\[ S : z = -x^4 - y^4 \]
\[ R : 0 \leq x \leq 1, \ 0 \leq y \leq 1 \]
\[ F = z \hat{i} + x \hat{j} + y \hat{k} \]

\[ \int_C F \cdot dr = \int_C (\nabla \times F) \cdot \nabla g \ dA \]

\[ g = x^4 + y^4 + z \]
\[ \nabla g = 4x^3 \hat{i} + 4y^3 \hat{j} + \hat{k} \]
\[ \nabla \times F = \hat{i} + \hat{j} + \hat{k} \]

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C \int_R \left[ (\nabla \times F) \cdot \nabla g \right] dA = \int_0^1 \int_0^1 \left[ (\hat{i} + \hat{j} + \hat{k}) \cdot (4x^3 \hat{i} + 4y^3 \hat{j} + \hat{k}) \right] dy dx \]

\[ = \int_0^1 \int_0^1 (4x^3 + 4y^3 + 1) \ dy \ dx = \int_0^1 4x^3 y + y^4 + y \ \bigg|_0^1 \ dx = \int_0^1 (4x^3 + 2) \ dx \]

\[ = x^4 + 2x \bigg|_0^1 = 1 + 2 = 3 \]