FINDING LOCAL MAXIMUMS AND MINIMUMS

DEFINITION:

Let *p* be a point in the domain in *f*.

- The function *f* has a ***local minimum*** at *p* if *f*(*p*) is less than or equal to the values of *f* for points near *p*.

- The function *f* has a ***local maximum*** at *p* if *f*(*p*) is greater than or equal to the values of *f* for points near *p*.

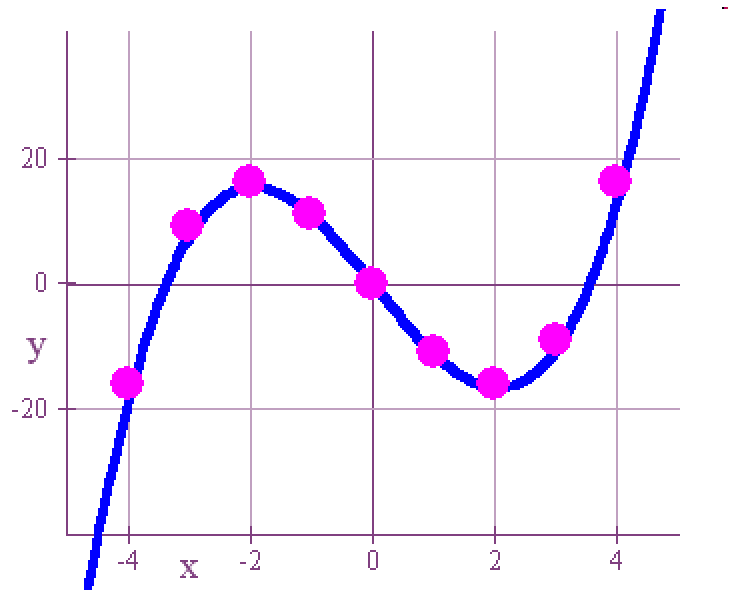
Recall the following:

THEOREM:

- If $df/dx > 0$ on an interval, then f is increasing on that interval.

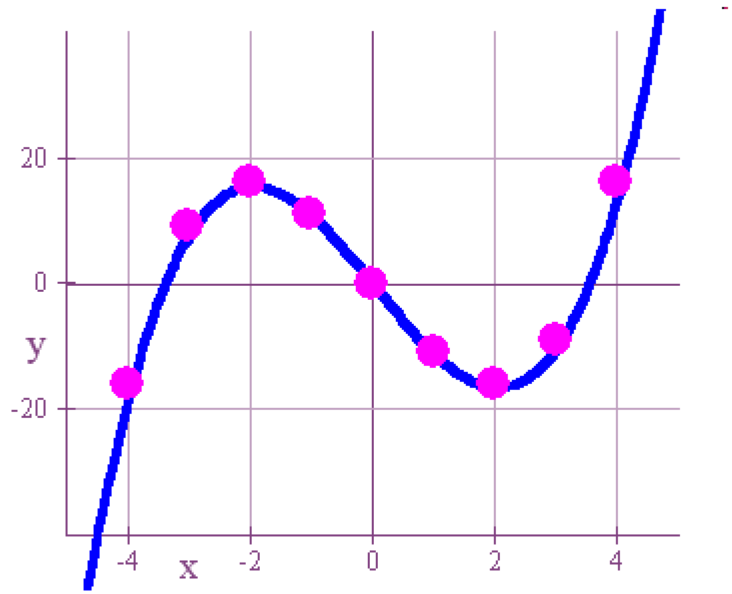
- If $df/dx < 0$ on an interval, then f is decreasing on that interval.

Consider the following function.



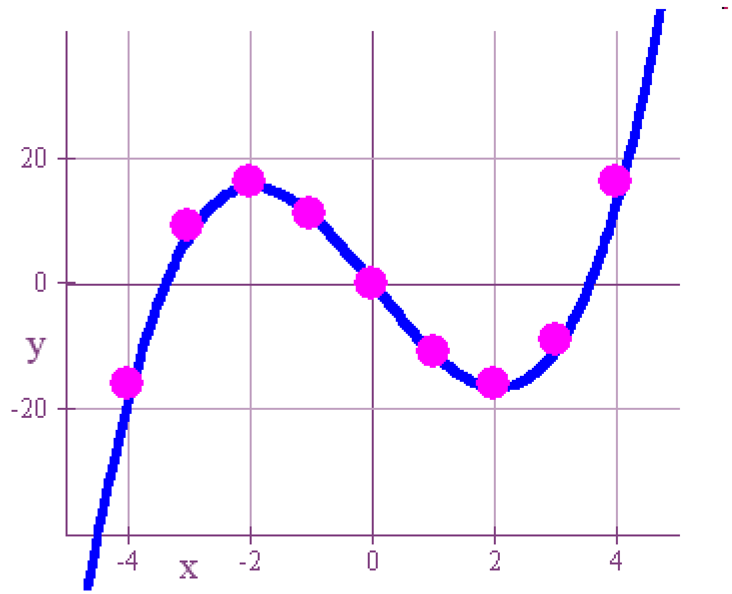
$$f(x) = x^3 - 12x$$

Where is *f*(*x*) increasing?



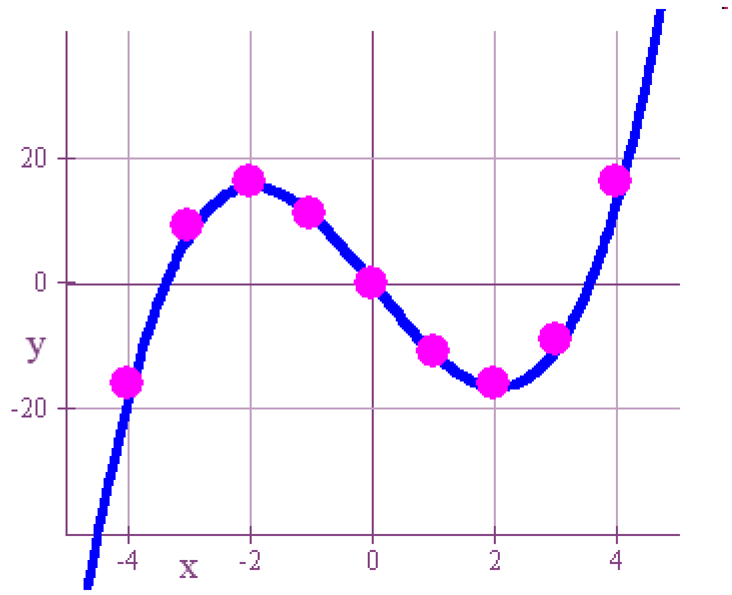
$$f(x) = x^3 - 12x$$

Where is *df/dx* positive?



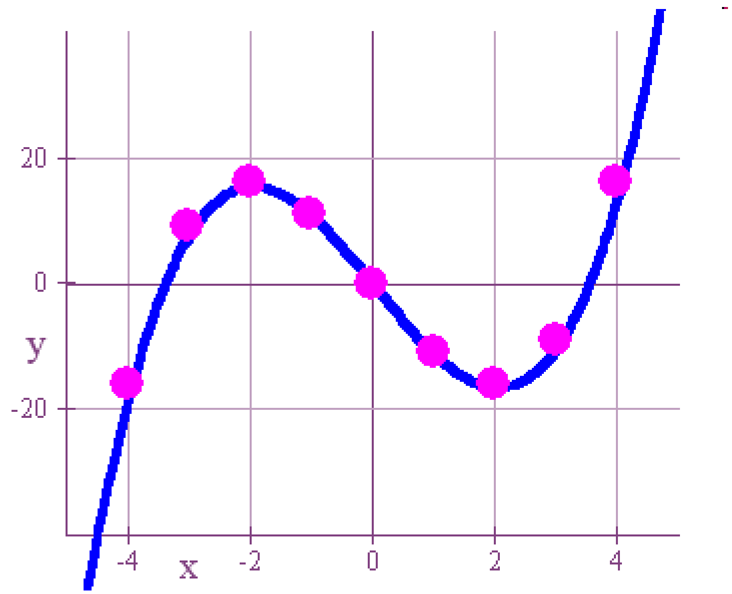
$$f(x) = x^3 - 12x$$

Where is *f*(*x*) decreasing?



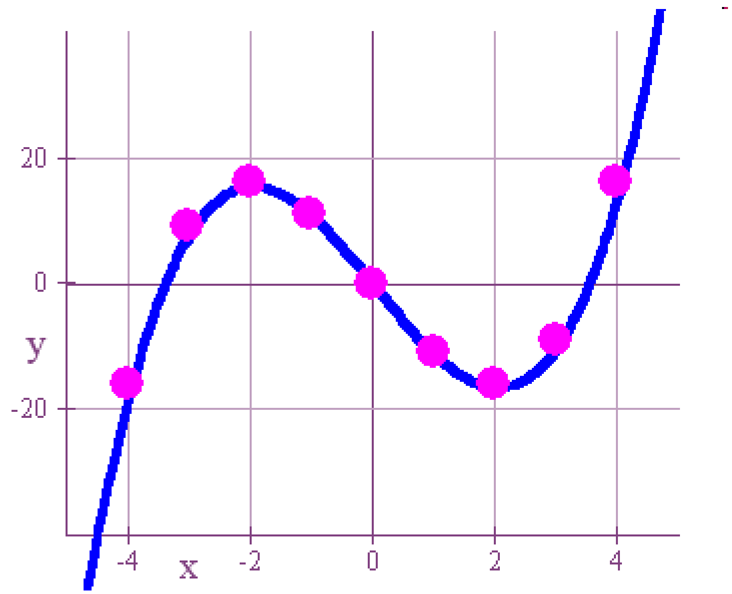
$$f(x) = x^3 - 12x$$

Where is *df/dx* negative?



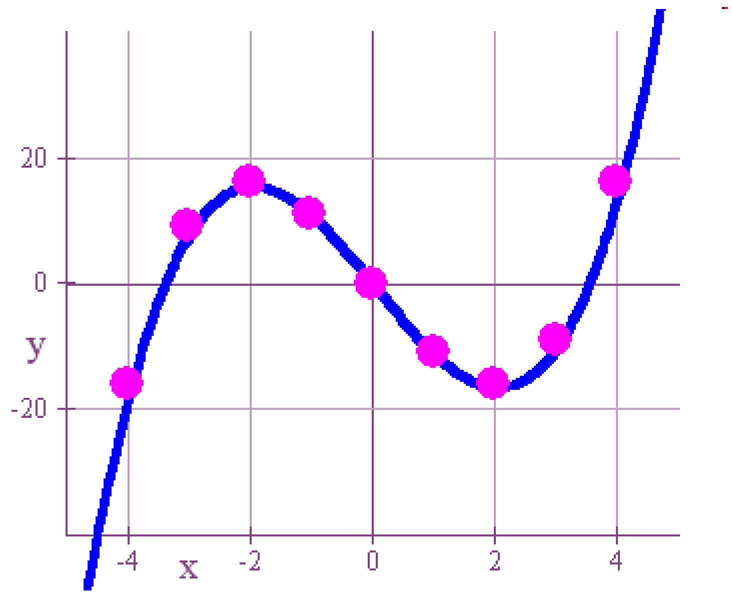
$$f(x) = x^3 - 12x$$

Where does *f*(*x*) have a local maximum?



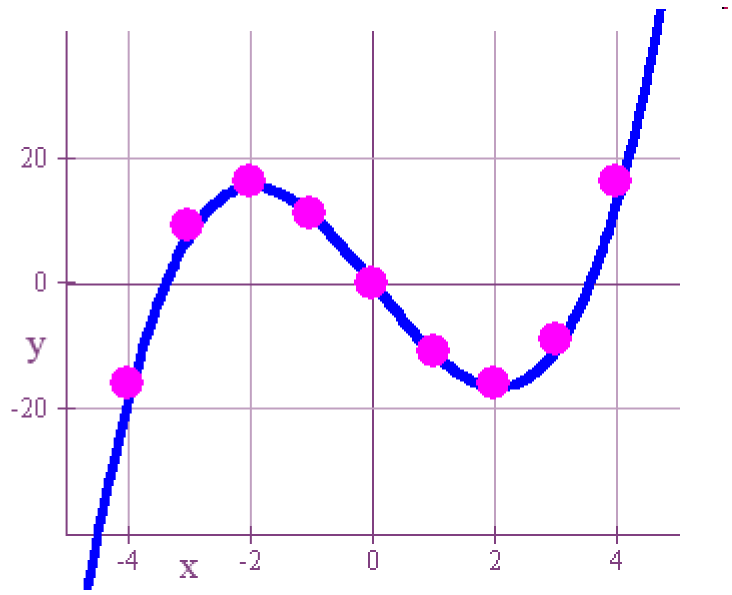
$$f(x) = x^3 - 12x$$

Where does *f*(*x*) have a local minimum?



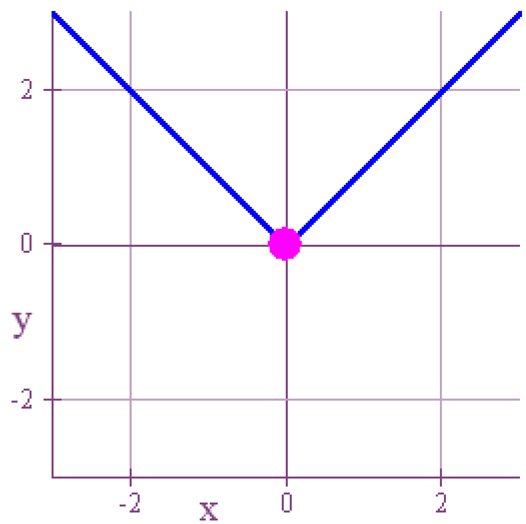
$$f(x) = x^3 - 12x$$

Where is *df/dx =0*?



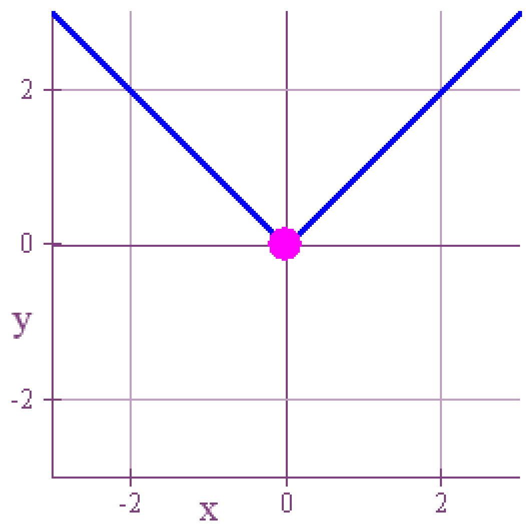
$$f(x) = x^3 - 12x$$

Now consider this function.



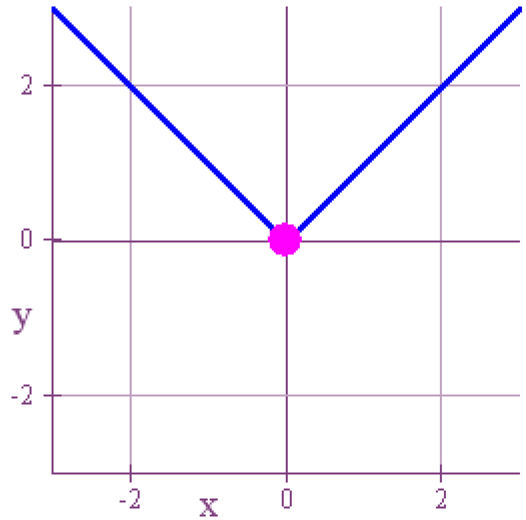
$$f(x) = |x|$$

Where is the local minimum?



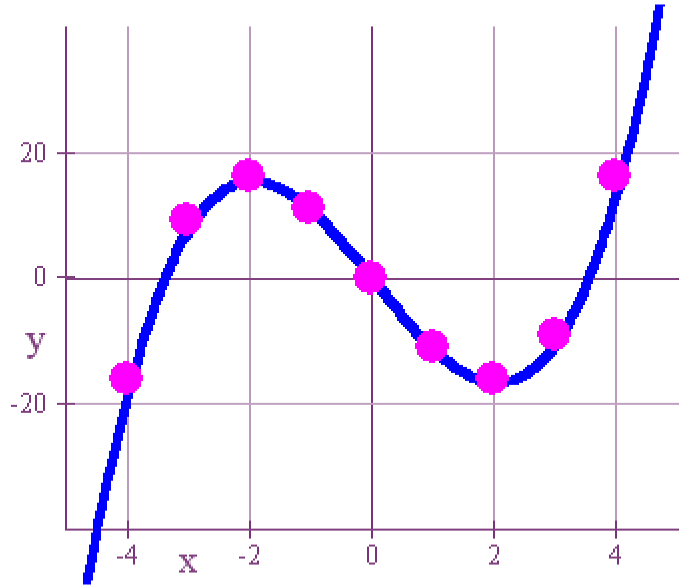
$$f(x) = |x|$$

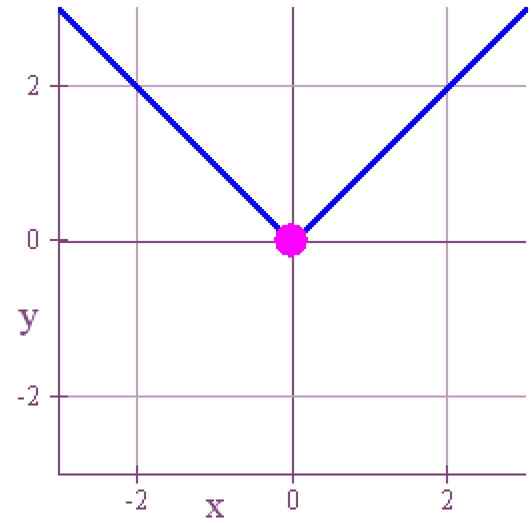
What is the value of the derivative at the point where the local minimum occurs?



$$f(x) = |x|$$

Can you make a conjecture regarding how we might use derivatives to help us locate local maximums and minimums?





Conjectures:

Local maximums and minimums tend to occur at places where the first derivative is zero or undefined.

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The derivative tends to change from positive to negative as we pass over a local maximum.

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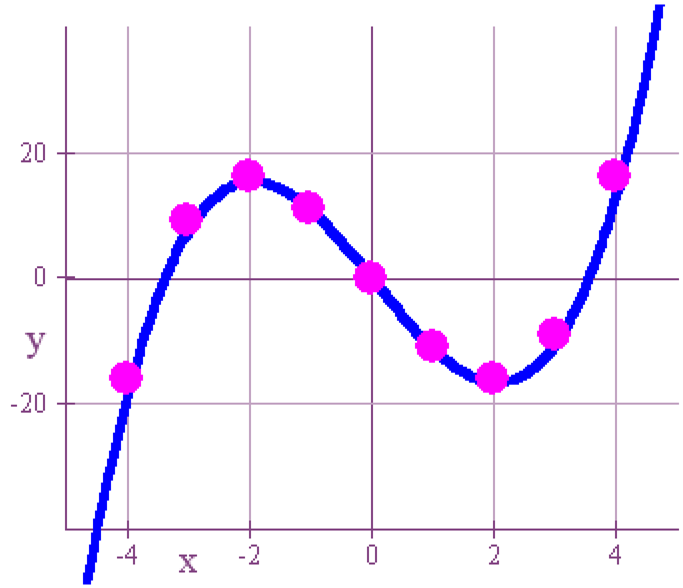
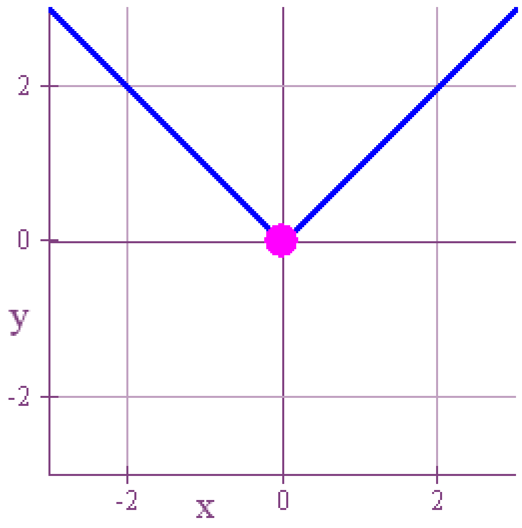
The derivative tends to change from positive to negative as we pass over a local maximum.

The derivative tends to change from negative to positive as we pass over a local minimum.

<u>Definition:</u> A point *p* in the domain of a function *f* is called a <u>*critical point*</u> if either the derivative at *p* is zero or the derivative at *p* is undefined.





Theorem: If $f(x)$ is defined on an interval and if $f(x)$ has a local maximum or minimum at a point $x = a$ which is not an endpoint of that interval, then $x = a$ is a *critical point*.

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In other words, for domain values that are not endpoints of an interval, the local maximums and minimums for our function will always occur at *critical points.*

Question: Will a *critical point* always result in a local maximum or minimum?

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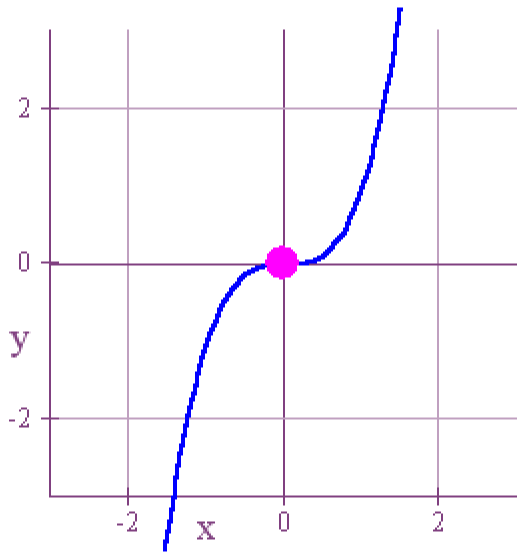
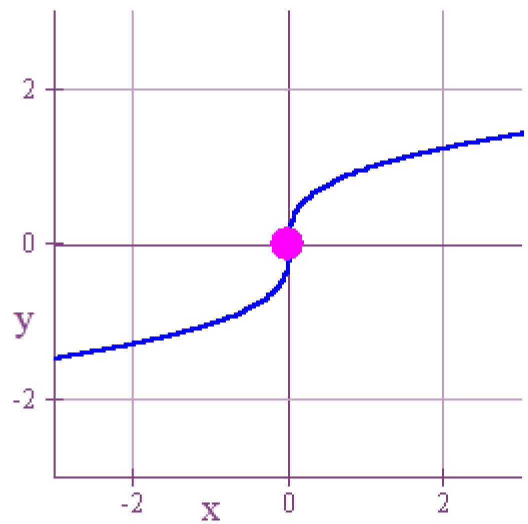
No! Consider the functions below.



$$f(x) = x^3$$



$$f(x) = \sqrt[3]{x}$$

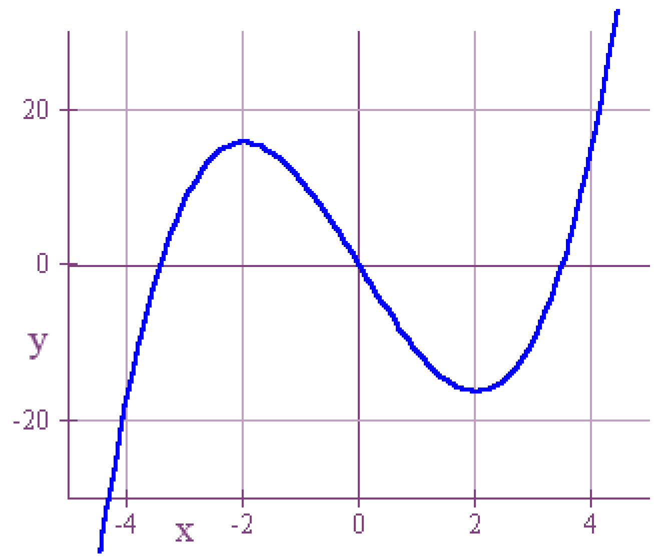
The First Derivative Test: Suppose *p* is a critical point of a continuous function *f*.

• If *df/dx* changes from negative to positive at *p*, then *f* has a local minimum at *p*.

• If *df/dx* changes from positive to negative at *p*, then *f* has a local maximum at *p*.

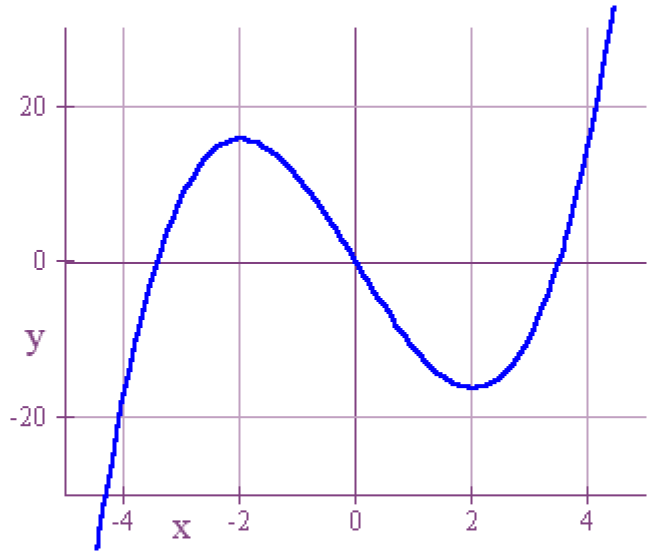
Example: The graph below suggests a local maximum at *x = -2* and a local minimum at *x = 2.*

$$f(x) = x^3 - 12x$$



Start by finding the critical points.

$$f(x) = x^3 - 12x$$

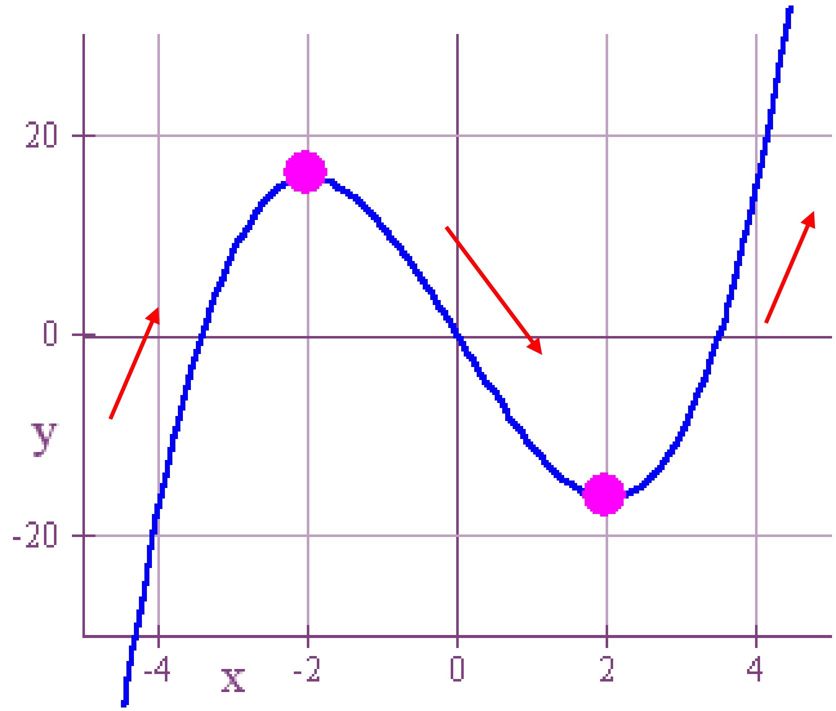


$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

$$f'(x) = 0 \Rightarrow x = -2 \text{ or } x = 2$$

Next, determine where the derivative is positive and where it's negative.

$$f(x) = x^3 - 12x$$



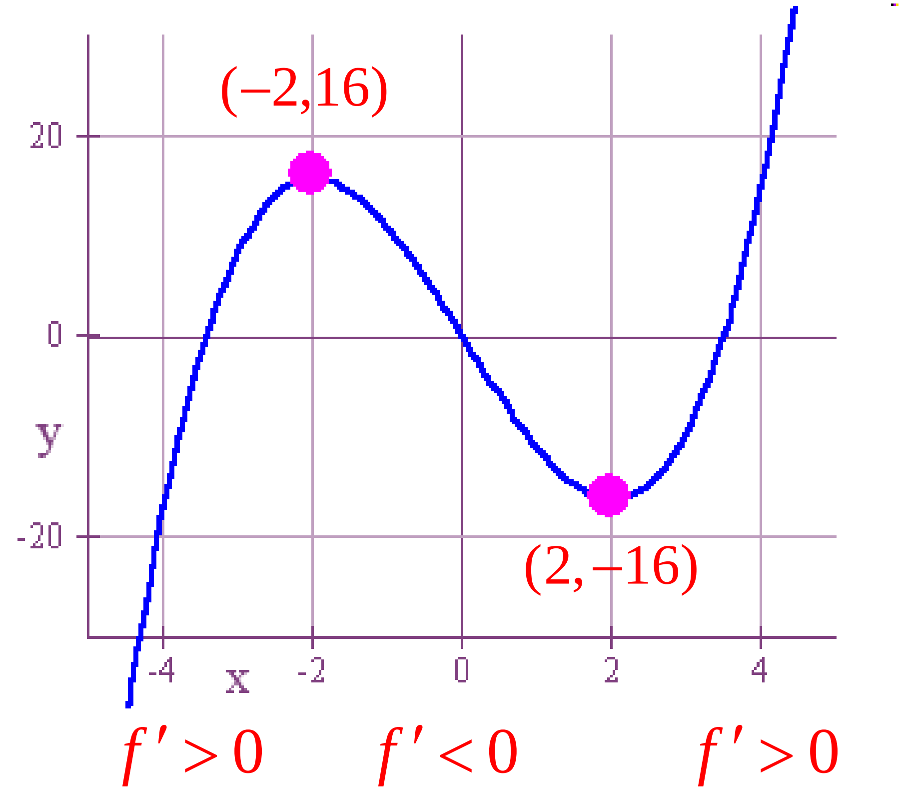
$f' > 0$ $f' < 0$ $f' > 0$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

$$f'(x) = 0 \Rightarrow x = -2 \text{ or } x = 2$$

And now, compute your local maximum and minimum.

$$f(x) = x^3 - 12x$$



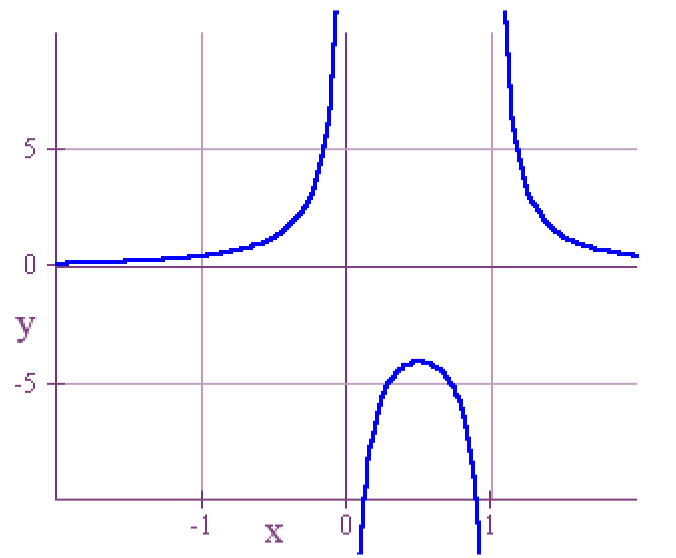
$f(-2) = 16$ is a local maximum

$f(2) = -16$ is a local minimum

Example: Find the *local extrema* for the function below.

$$f(x) = \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

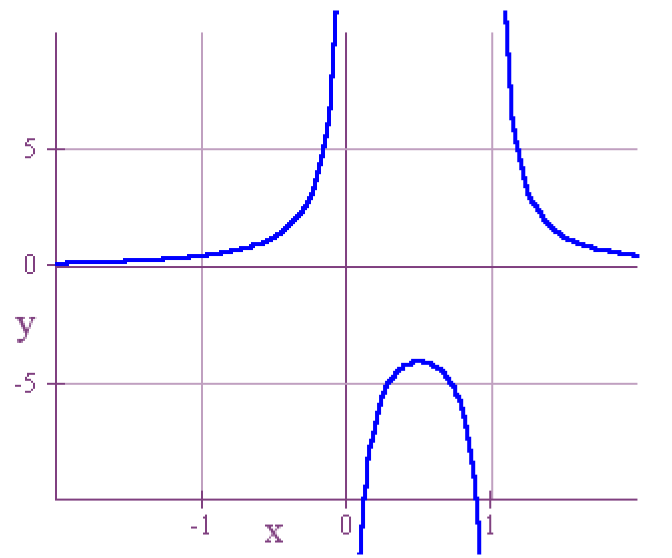
$$= (x^2 - x)^{-1}$$



The graph suggests a local maximum occurs in the interval (0,1).

$$f(x) = \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

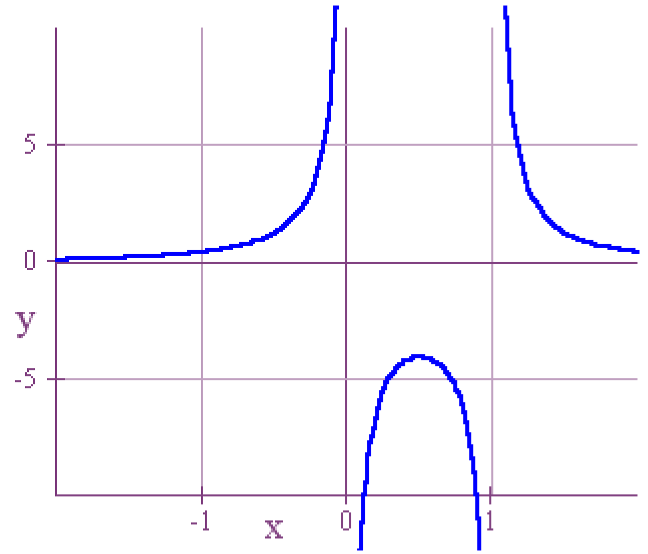
$$= (x^2 - x)^{-1}$$



Now, find the derivative.

$$f(x) = \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

$$= (x^2 - x)^{-1}$$



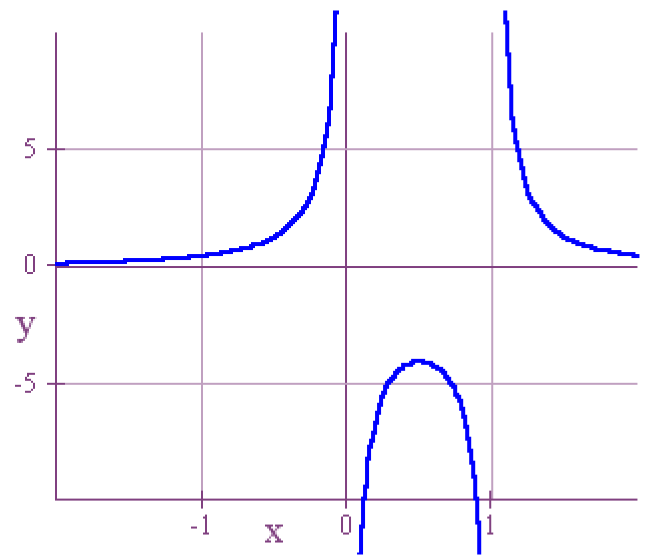
$$f'(x) = -(x^2 - x)^{-2}(2x-1)$$

$$= -\frac{2x-1}{(x^2-x)^2}$$

And finally, find the critical point.

$$f(x) = \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

$$= (x^2 - x)^{-1}$$



$$f'(x) = -(x^2 - x)^{-2}(2x-1)$$

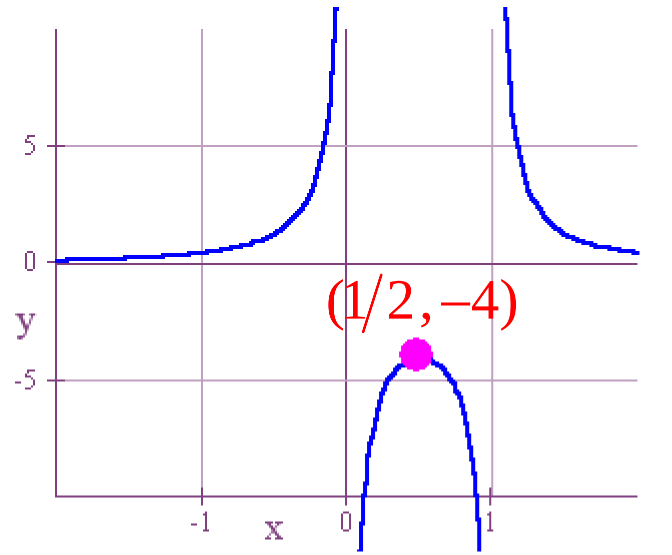
$$= -\frac{2x-1}{(x^2-x)^2}$$

$$f'(x) = 0 \Rightarrow x = 1/2$$

The First Derivative Test now verifies that we have a local maximum at *x* = *1/2*.

$$f(x) = \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

$$= (x^2 - x)^{-1}$$



$$f'(x) = -(x^2 - x)^{-2}(2x-1)$$

$$= -\frac{2x-1}{(x^2-x)^2}$$

$$f'(x) = 0 \Rightarrow x = 1/2$$

$$f(1/2) = -4$$

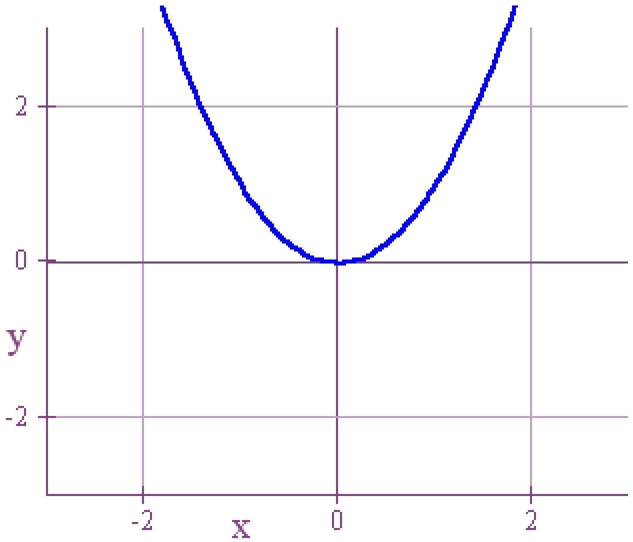
Now let’s look at another way to find local maximums and minimums.

Recall the following:

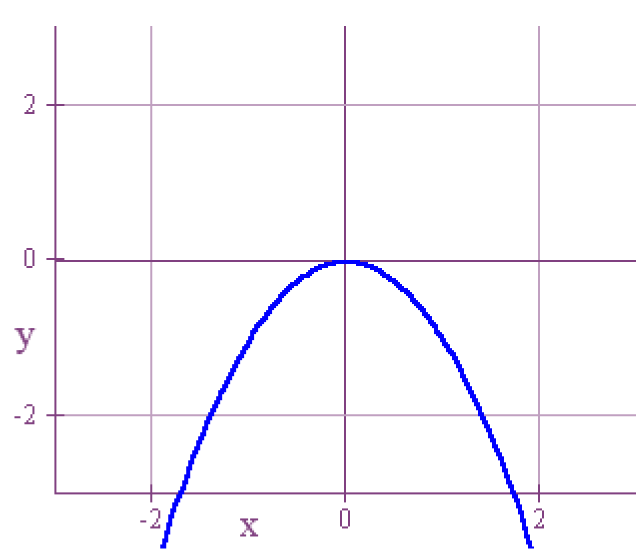
THEOREM:

- If $df^2/dx^2 > 0$ on an interval, then f is concave up on that interval.
- If $df^2/dx^2 < 0$ on an interval, then f is concave down on that interval.

Now look at the graphs below and make a conjecture at how we might use the second derivative in conjunction with the first derivative to help us locate local extrema.



$$f(x) = x^2$$
$$f'(0) = 0$$
$$f''(0) = 2$$



$$f(x) = -x^2$$
$$f'(0) = 0$$
$$f''(0) = -2$$

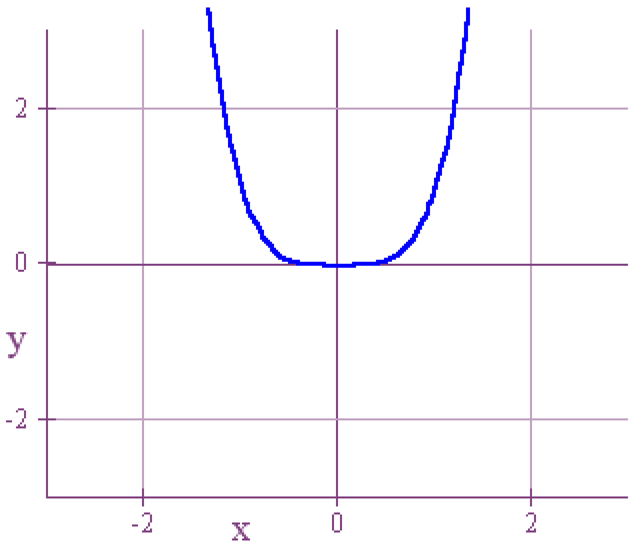
Conjecture:

If at $x = p$, the first derivative is zero and the second derivative is positive, then we have a ***local minimum*** at $x = p$.

If at $x = p$, the first derivative is zero and the second derivative is negative, then we have a ***local maximum*** at $x = p$.

Question:

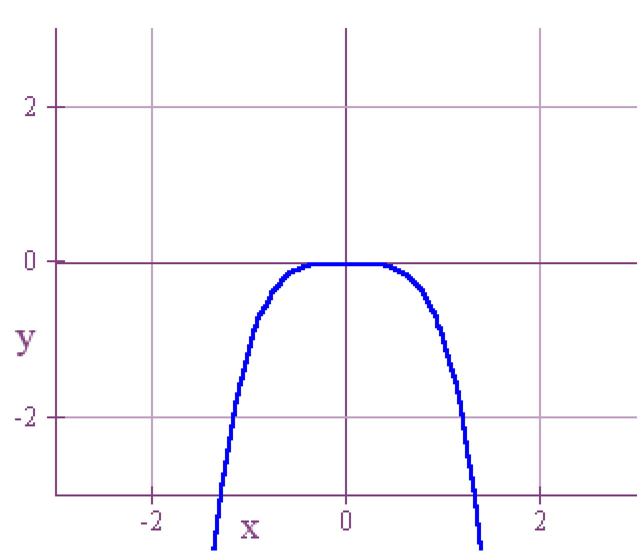
Do we know anything if both the first and second derivative are zero?



$$f(x) = x^4$$

$$f'(0) = 0$$

$$f''(0) = 0$$



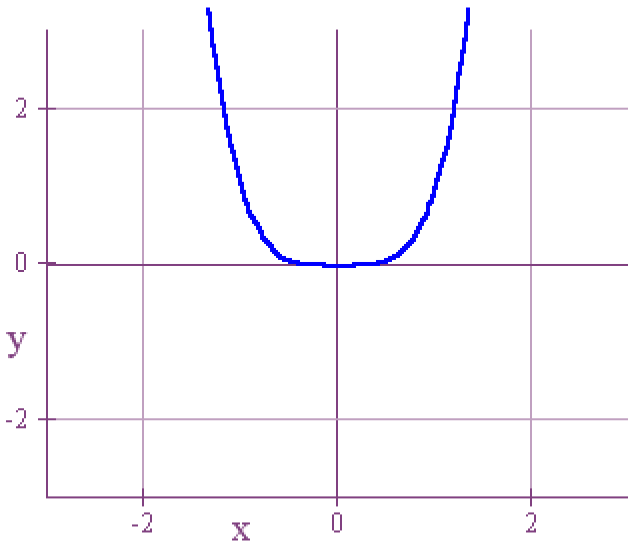
$$f(x) = -x^4$$

$$f'(0) = 0$$

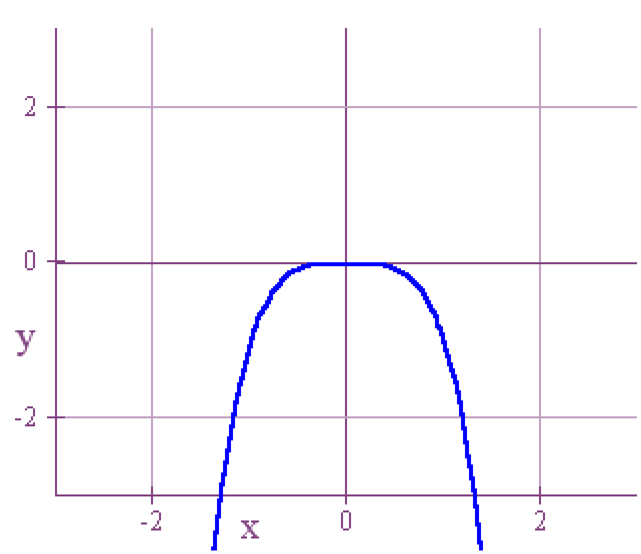
$$f''(0) = 0$$

Question:

Do we know anything if both the first and second derivative are zero? NO!



$$f(x) = x^4$$
$$f'(0) = 0$$
$$f''(0) = 0$$



$$f(x) = -x^4$$
$$f'(0) = 0$$
$$f''(0) = 0$$

The Second Derivative Test:

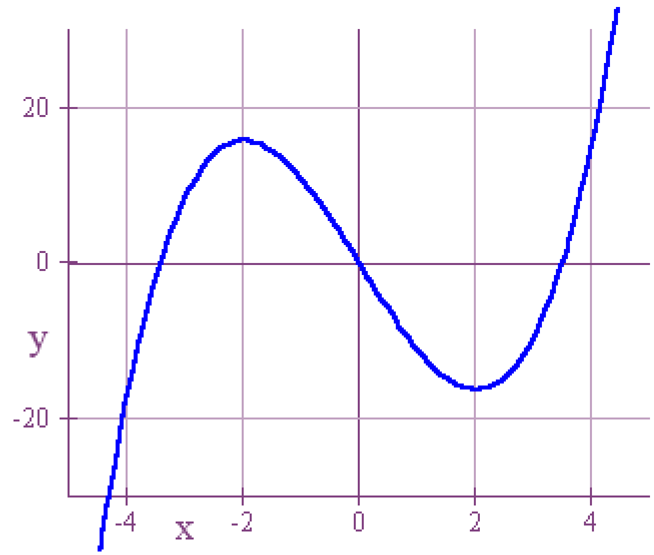
If at $x = p$, the first derivative is zero and the second derivative is positive, then we have a ***local minimum*** at $x = p$.

If at $x = p$, the first derivative is zero and the second derivative is negative, then we have a ***local maximum*** at $x = p$.

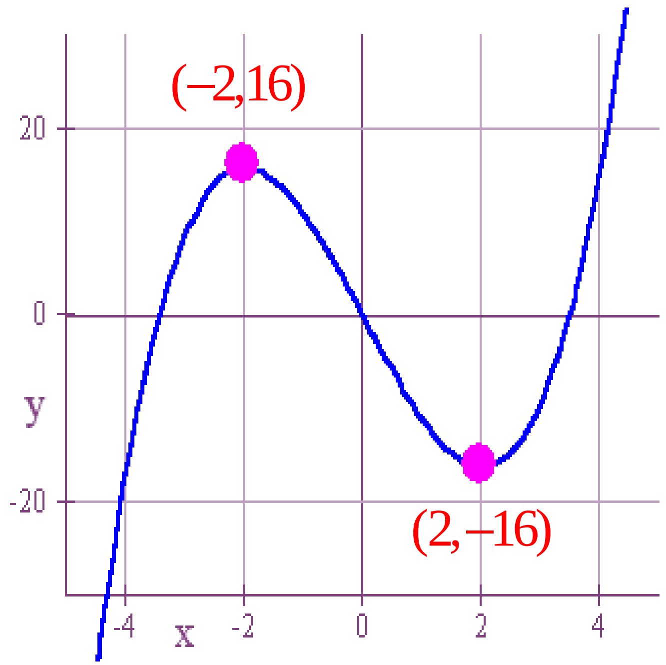
If at $x = p$, both the first and second derivaties are zero, then we know nothing.

Example: Use the second derivative test to find the local extrema.

$$f(x) = x^3 - 12x$$



$$f(x) = x^3 - 12x$$



$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

$$f'(x) = 0 \Rightarrow x = -2 \text{ or } x = 2$$

$$f''(x) = 6x$$

$$f''(-2) = -12 < 0 \text{ and } f''(2) = 12 > 0$$

$$f(-2) = 16 \text{ is a local maximum}$$

$$f(2) = -16 \text{ is a local minimum}$$

Inflection Points:

A point at which the graph of a function changes concavity is called an ***inflection point***. If *df/dx* is continuous, then:

A point *p* where the second derivative is zero or undefined is a possible inflection point.

To test whether *p* is a inflection point, check whether the second derivative changes sign at *p*.

Explain why the following is true.

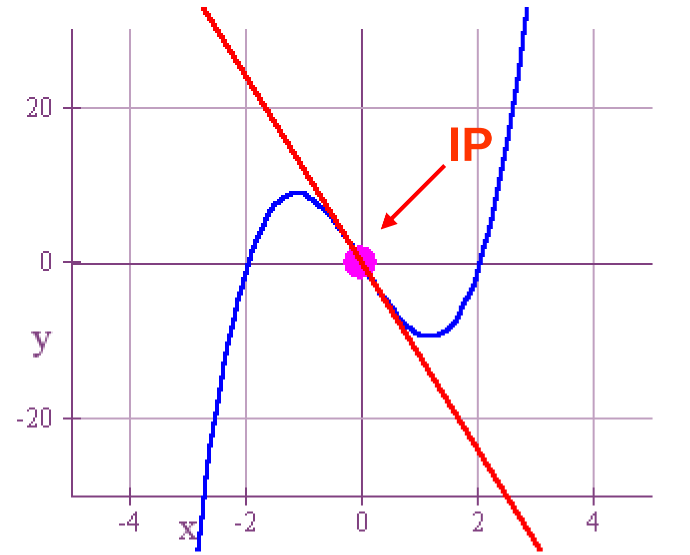
Suppose $f(x)$ has a continuous derivative. If the second derivative changes sign at p, then $f(x)$ has an inflection point at p, and the first derivative has a local minimum or a local maximum at p.

Example:.

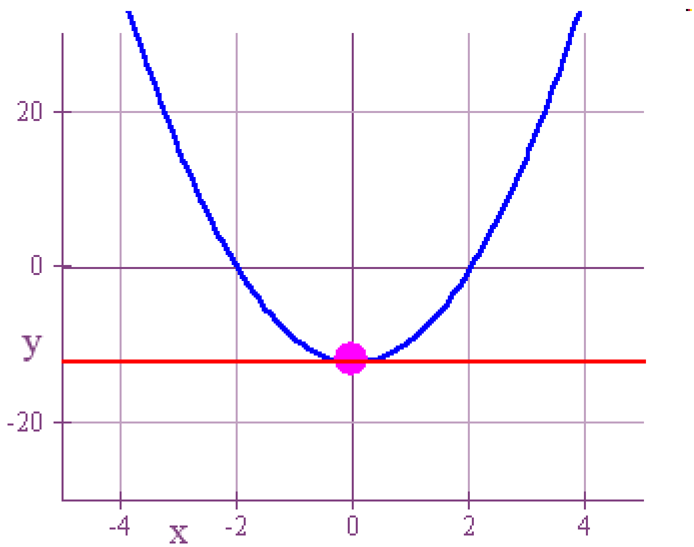
$$f(x) = x^3 - 12x$$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x$$



$f'' < 0$ $\quad f'' = 0$ $\quad f'' > 0$



$$f'(x) = 3x^{12} - 12$$

And finally, let's look at problems 12, 28, and 34 in the exercises for 4.1 (pages 182-183).

Section 4.1, problem 12:

Find all critical points of $f(x) = 10.2x^2 e^{-0.4x}$.

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$$f'(x) = -4.08x^2 e^{-0.4x} + 20.4xe^{-0.4x} = -4.08xe^{-0.4x}(x-5)$$

$$f'(x) = 0 \Rightarrow ?$$

Section 4.1, problem 12:

Find all critical points of $f(x) = 10.2x^2e^{-0.4x}$.

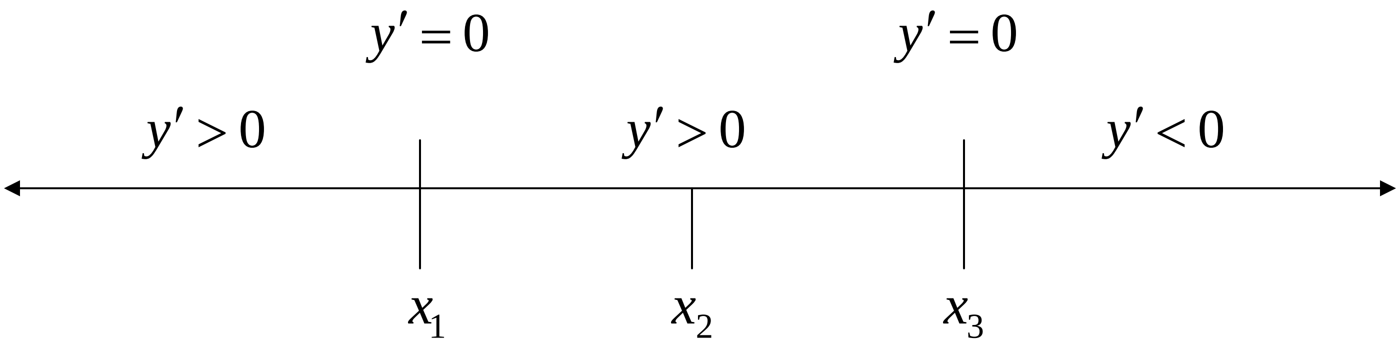
$$f'(x) = -4.08x^2e^{-0.4x} + 20.4xe^{-0.4x} = -4.08xe^{-0.4x}(x-5)$$

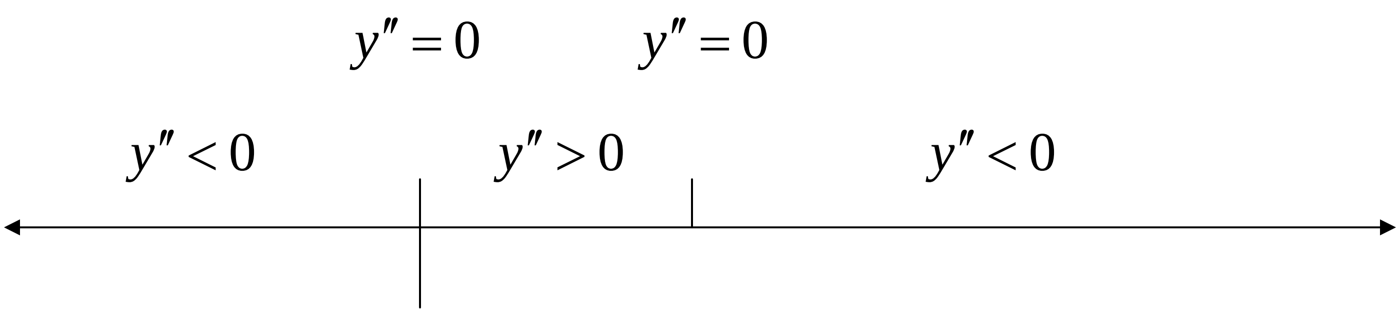
$$f'(x) = 0 \Rightarrow x = 0 \text{ or } x = 5$$

Section 4.1, problem 28:

Sketch a possible graph of $y = f(x)$.

Assime that the function is defined and continuous for all real x.





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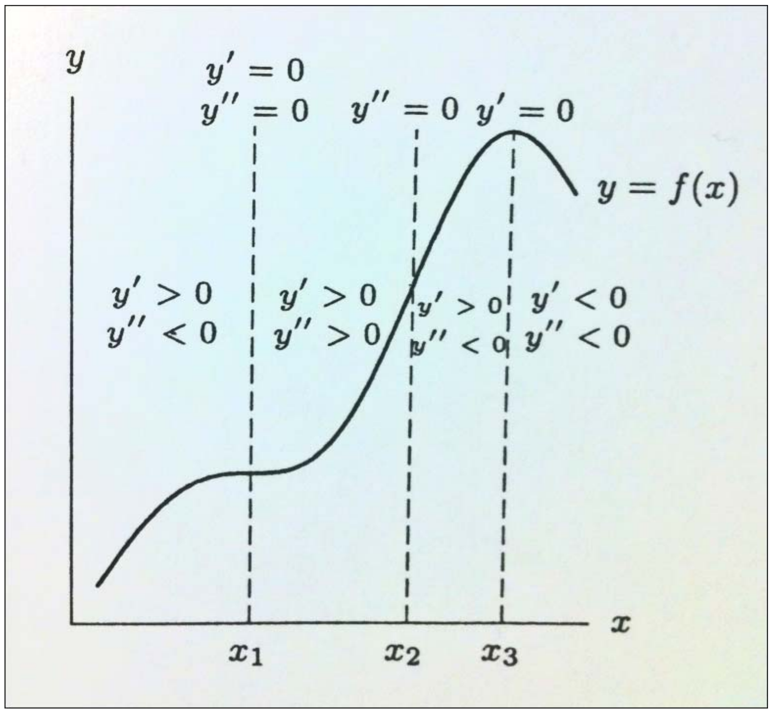

Section 4.1, problem 34:

Water is flowing at a constant rate (i.e. constant volume per unit of time) into a cylindrical container standing vertically. Sketch a graph showing the depth of water against time.

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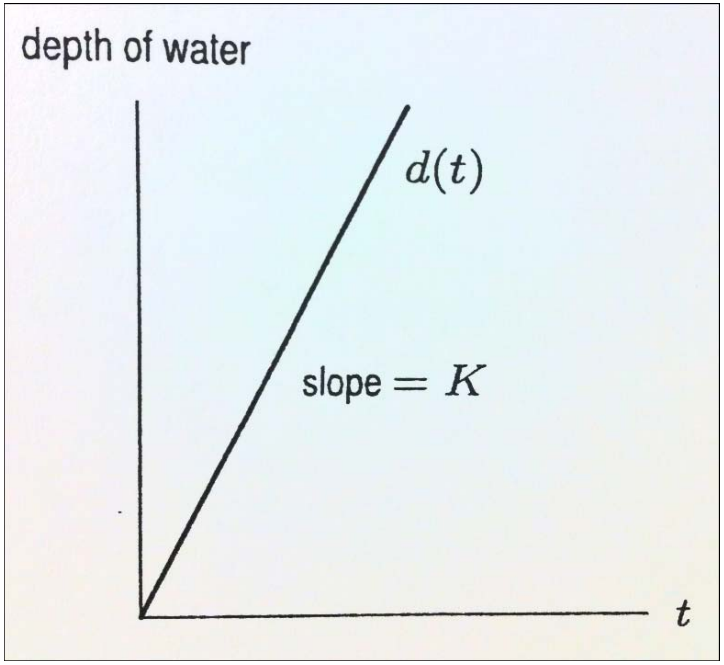

Section 4.1, problem 34:

Water is flowing at a constant rate (i.e. constant volume per unit of time) into a cone-shaped container standing on its point. Sketch a graph showing the depth of water against time.

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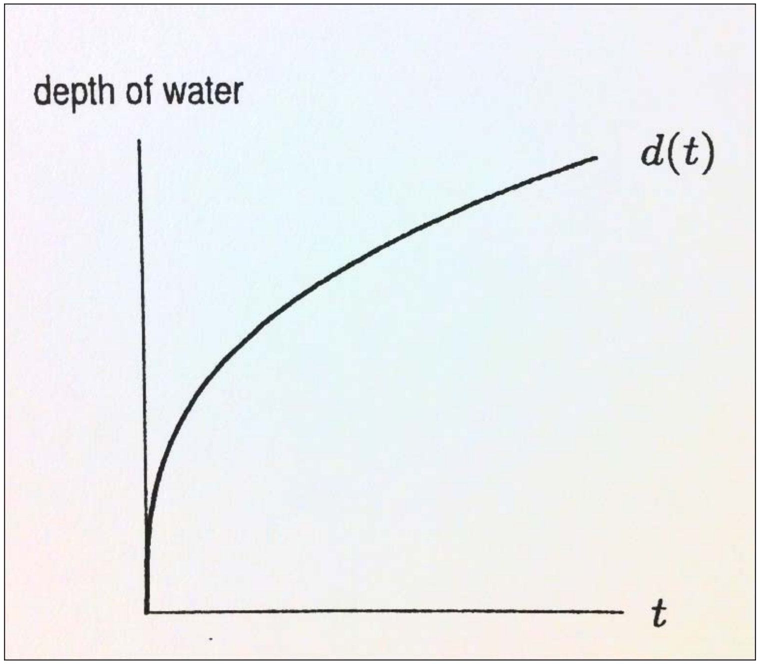


"That's
all
folks!"